

# STOCHASTIC INTEGRAL EQUATIONS FOR WALSH SEMIMARTINGALES <sup>\*</sup>

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## Abstract

We construct planar semimartingales that include the Walsh Brownian motion as a special case, and derive Harrison-Shepp-type equations and a change-of-variable formula in the spirit of Freidlin-Sheu for these so-called “Walsh semimartingales”. We examine the solvability of the resulting system of stochastic integral equations. In appropriate Markovian settings we study two types of connections to martingale problems, questions of uniqueness in distribution for such processes, and a few examples.

*Key Words and Phrases:* Skew and Walsh Brownian motions, spider and Walsh semimartingales, Skorokhod reflection, planar skew unfolding, Harrison-Shepp equations, Freidlin-Sheu formula, martingale problems, local time.

*AMS 2000 Subject Classifications:* Primary, 60G42; secondary, 60H10.

## 1 Introduction and Summary

We consider the following questions: *What is a two-dimensional analogue of the skew Brownian motion on the real line? If such a process exists, what is the corresponding stochastic integral equation that realizes its construction and describes its dynamics? Are there more general planar semimartingales with similar skew-unfolding-type structure?*

In order to answer the first question, WALSH (1978) introduced a singular planar diffusion with these properties. This diffusion is known now as the WALSH Brownian motion. In its description by BARLOW, PITMAN & YOR (1989), “started at a point in the plane away from the origin, this process moves like a standard Brownian motion along the ray joining the starting point and the origin  $\mathbf{0}$ , until it reaches  $\mathbf{0}$ . Then it is kicked away from  $\mathbf{0}$  by an entrance law that makes the radial part of the diffusion a reflecting Brownian motion, while randomizing the angular part”. The WALSH Brownian motion has been generalized to the so-called spider martingales, and has been studied by several researchers (among them BARLOW, PITMAN

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& YOR (1989), TSIREL'SON (1997), WATANABE (1999), EVANS & SOWERS (2003), PICARD (2005), FREIDLIN & SHEU (2000), MANSUY & YOR (2006), HAJRI (2011), FITZSIMMONS & KUTER (2014), HAJRI & TOUHAMI (2014), CHEN & FUKUSHIMA (2015)). In this paper we construct a family of planar semimartingales that includes the spider martingales and the WALSH Brownian motion as special cases.

There are several constructions of WALSH's Brownian motions in terms of resolvents, infinitesimal generators, semigroups, and excursion theory. Our approach in this paper can be thought of as a bridge between excursion theory and stochastic integral equations, via the folding and unfolding of semimartingales. It is also an attempt to study higher-dimensional analogues of the skew-TANAKA equation, and the semimartingale properties of planar processes that hit points.

*Preview:* We provide in Section 2 a system of stochastic equations (2.12) that these planar semimartingales satisfy. This system is a two-dimensional analogue of the equation introduced by HARRISON & SHEPP (1981) for the skew Brownian motion, and answers the second and third questions stated above. Based on this integral-equation description, we develop in Sections 3 and 4 a stochastic calculus, and establish a FREIDLIN-SHEU type change-of-variable formula (4.5), for such WALSH semimartingales. We also develop a condition (3.7) closely analogous to that of HARRISON & SHEPP (1981), for the solvability of our system of equations (2.12). In Section 5 we examine by the method of PROKAJ (2009) this two-dimensional HARRISON-SHEPP equation driven by a continuous semimartingale, as in ICHIBA & KARATZAS (2014).

Pathwise uniqueness fails for the system of (2.12). For the ITÔ diffusion case, we recast this system in the form (6.4), and study its connections to an appropriate martingale problem in Sections 6 and 8. The well-posedness of this martingale problem, in the form of conditions under which a weak solution exists for the system of (6.4) and is unique in distribution, is based on the stochastic calculus of Section 4.

The WALSH Brownian motion constructed via the FELLER semigroup, is then shown in Section 7 to be a special case of our “WALSH diffusion” framework. Another type of connection to martingale problems is established in Section 9, allowing us to show that WALSH diffusions are the only time-homogeneous and strongly Markovian solutions of the system (6.4). A notable difference from the HARRISON-SHEPP equation is also given there; whereas in Section 10 we study additional examples. Some auxiliary results and proofs are provided in the appendices, Sections 11 and 12.

## 2 The Setting and Results

On a filtered probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ ,  $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}(t)\}_{0 \leq t < \infty}$  that satisfies the “usual conditions” of right-continuity and augmentation by null sets, we consider a real-valued, continuous semimartingale

$$U(t) = M(t) + V(t), \quad 0 \leq t < \infty. \quad (2.1)$$

Here  $M(\cdot)$  is a continuous local martingale and  $V(\cdot)$  has finite variation on compact intervals; we assume that the initial position  $U(0) \geq 0$  is a given real number. We denote by

$$S(t) := U(t) + \Lambda(t), \quad \text{where} \quad \Lambda(t) = \max_{0 \leq s \leq t} (-U(s))^+, \quad 0 \leq t < \infty, \quad (2.2)$$

the SKOROKHOD reflection (or “folding”) of  $U(\cdot)$ ; see, for instance, section 3.6 in KARATZAS & SHREVE (1991) for relevant theory. In particular, the continuous, increasing process  $\Lambda(\cdot)$  is flat off the zero set

$$\mathfrak{Z} := \{0 \leq t < \infty : S(t) = 0\}. \quad (2.3)$$

We shall impose the “non-stickiness” condition

$$\text{Leb}(\mathfrak{Z}) \equiv \int_0^\infty \mathbf{1}_{\{S(t)=0\}} dt = 0. \quad (2.4)$$

Let us recall the (*right*) *local time*  $L^\Xi(\cdot)$  accumulated at the origin during the time-interval  $[0, T]$  by a generic one-dimensional continuous semimartingale  $\Xi(\cdot)$ , namely

$$L^\Xi(T) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^T \mathbf{1}_{\{0 \leq \Xi(t) < \varepsilon\}} d\langle \Xi \rangle(t), \quad 0 \leq T < \infty. \quad (2.5)$$

From (2.2) and by analogy with Lemma 3.1.5 in PICARD (2005), we have the ITÔ-TANAKA-type equation

$$S(\cdot) = S(0) + \int_0^\cdot \mathbf{1}_{\{S(t) > 0\}} dU(t) + L^S(\cdot), \quad S(0) = U(0) \geq 0. \quad (2.6)$$

On the other hand, the theory of semimartingale local time (e.g., section 3.7 in KARATZAS & SHREVE (1991)) gives the properties

$$\int_0^\infty \mathbf{1}_{\{S(t)=0\}} d\langle S \rangle(t) = 0, \quad L^S(\cdot) = \int_0^\cdot \mathbf{1}_{\{S(t)=0\}} dS(t). \quad (2.7)$$

## 2.1 The Main Result

Theorem 2.1 below is the first key result of this paper. It produces a planar “skew-unfolding”  $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$  for the folding  $S(\cdot)$  of the given continuous semimartingale  $U(\cdot)$ . This planar “skew-unfolded” process has radial part  $\|X(\cdot)\| = S(\cdot)$ , and its motion away from the origin follows the one-dimensional dynamics of  $S(\cdot)$  along rays emanating from the origin. Once at the origin, the process  $X(\cdot)$  chooses the next ray for its voyage (according to the dynamics of  $S(\cdot)$ ) independently of its past history and in a random fashion, according to a given probability measure on the collection of angles in  $[0, 2\pi)$ . Whenever  $S(\cdot)$  is a reflecting Brownian motion or, more generally, a reflecting diffusion, these one-dimensional dynamics away from the origin are of course diffusive.

In order to describe this skew-unfolding with some detail and rigor, we shall need appropriate notation. Let us consider the unit circumference

$$\mathfrak{S} := \{(z_1, z_2)' : z_1^2 + z_2^2 = 1\}.$$

Here and throughout the paper, vectors are columns and the superscript  $'$  denotes transposition. For every point  $x := (x_1, x_2)' \in \mathbb{R}^2$  we introduce the mapping  $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2) : \mathbb{R}^2 \rightarrow \mathfrak{S} \cup \{\mathbf{0}\}$  via  $\mathfrak{f}(\mathbf{0}) := \mathbf{0}$  and

$$\mathfrak{f}(x) := \frac{x}{\|x\|} = (\cos(\arg(x)), \sin(\arg(x)))'; \quad x \in E := \mathbb{R}^2 \setminus \{\mathbf{0}\} \quad (2.8)$$

with the notation  $\mathbf{0} := (0, 0)'$  and with  $\arg(x) \in [0, 2\pi)$  denoting the *argument of the vector*  $x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$  in its polar coördinates. We fix a probability measure  $\mu$  on the collection  $\mathcal{B}(\mathfrak{S})$  of Borel subsets of the unit circumference  $\mathfrak{S}$ , and consider also its expression

$$\nu(d\theta) := \mu(dz), \quad z = (\cos(\theta), \sin(\theta))' \in \mathfrak{S}, \quad \theta \in [0, 2\pi) \quad (2.9)$$

in polar coördinates. We introduce the real constants

$$\alpha_i^{(\pm)} := \int_{\mathfrak{S}} (\mathfrak{f}_i(z))^{\pm} \mu(dz), \quad \gamma_i := \alpha_i^{(+)} - \alpha_i^{(-)} = \int_{\mathfrak{S}} \mathfrak{f}_i(z) \mu(dz), \quad i = 1, 2 \quad (2.10)$$

as well as the vector on the unit disc

$$\gamma := (\gamma_1, \gamma_2)' = \left( \int_0^{2\pi} \cos(\theta) \nu(d\theta), \int_0^{2\pi} \sin(\theta) \nu(d\theta) \right)'. \quad (2.11)$$

Finally, we fix a vector  $x := (x_1, x_2)' \in \mathbb{R}^2$  with  $x_i = \mathfrak{f}_i(x) S(0)$ ,  $i = 1, 2$ .

**Theorem 2.1. Construction of WALSH Semimartingales:** Consider the SKOROKHOD reflection  $S(\cdot)$  of the continuous semimartingale  $U(\cdot)$  as in (2.1) – (2.4), and fix a vector  $\mathbf{x} = (x_1, x_2)' \in \mathbb{R}^2$  as above.

On a suitable enlargement  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $\mathbb{F} := \{\mathcal{F}(t)\}_{0 \leq t < \infty}$  of the filtered probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ ,  $\tilde{\mathbb{F}}$  with a measure preserving map  $\pi : \Omega \rightarrow \tilde{\Omega}$ , there exists a planar continuous semimartingale  $X(\cdot) := (X_1(\cdot), X_2(\cdot))'$  which solves the system of stochastic integral equations

$$X_i(T) = x_i + \int_0^T f_i(X(t)) dS(t) + \left( \alpha_i^{(+)} - \alpha_i^{(-)} \right) L^S(T), \quad 0 \leq T < \infty \quad (2.12)$$

for  $i = 1, 2$  and whose radial part is

$$\|X(\cdot)\| := \sqrt{X_1^2(\cdot) + X_2^2(\cdot)} = S(\cdot). \quad (2.13)$$

This continuous semimartingale  $X(\cdot) := (X_1(\cdot), X_2(\cdot))'$  has the following properties:

(i) With  $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$  and  $\tau(s) := \inf \{t > s : X(t) = \mathbf{0}\}$  the first time it reaches the origin after time  $s \geq 0$ , this process  $X(\cdot)$  satisfies for every  $s \in (0, \infty)$ ,  $B \in \mathcal{B}(\mathfrak{S})$  and for Lebesgue almost every  $t \in (0, \infty)$  the properties

$$f(X(s)) = f(\mathbf{x}), \quad \mathbb{P} - \text{a.e. on } \{\tau(0) > s\}, \quad (2.14)$$

$$\mathbb{P}(f(X(\tau(s) + t)) \in B \mid \mathcal{F}^X(\tau(s))) = \boldsymbol{\mu}(B), \quad \mathbb{P} - \text{a.e. on } \{\tau(s) < \infty\}. \quad (2.15)$$

(ii) The local times at the origin of the component processes  $X_i(\cdot)$  are given as

$$L^{X_i}(\cdot) \equiv \alpha_i^{(+)} L^{\|X\|}(\cdot), \quad i = 1, 2 \quad (2.16)$$

and are thus flat off the random set  $\mathfrak{Z}$  in (2.3) which has zero LEBESGUE measure by (2.4); in particular,

$$\int_0^\infty \mathbf{1}_{\{X(t) = \mathbf{0}\}} dt \equiv 0. \quad (2.17)$$

(iii) Finally, for every  $A \in \mathcal{B}([0, 2\pi))$ , the semimartingale local time at the origin of the “thinned” process  $R^A(\cdot) := \|X(\cdot)\| \cdot \mathbf{1}_A(\arg(X(\cdot)))$  is given by

$$L^{R^A}(\cdot) \equiv \boldsymbol{\nu}(A) L^{\|X\|}(\cdot). \quad (2.18)$$

**Terminology 2.1.** The process  $X(\cdot)$  constructed in the above Theorem can be thought of as a planar skew-unfolding of the SKOROKHOD reflection  $S(\cdot)$  of the driving continuous semimartingale  $U(\cdot)$ ; we shall call it a WALSH semimartingale with “driver”  $U(\cdot)$  (and “folded driver”  $S(\cdot)$ ). We shall call  $X(\cdot)$  a WALSH diffusion, whenever the folded driver  $S(\cdot)$  is an ITÔ diffusion with reflection at the origin.

With the family of increasing processes  $L^{R^A}(\cdot)$ ,  $A \in \mathcal{B}([0, 2\pi))$  in (2.18), we shall find it conceptually convenient to associate a random product measure  $\boldsymbol{\Lambda}^X(dt, d\theta)$  on  $[0, \infty) \times [0, 2\pi)$  via

$$\boldsymbol{\Lambda}^X([0, t) \times A) := L^{R^A}(t) = \boldsymbol{\nu}(A) L^{\|X\|}(t); \quad 0 \leq t < \infty, \quad A \in \mathcal{B}([0, 2\pi)). \quad (2.19)$$

### 3 Discussion and Ramifications

An intuitive interpretation of the stochastic integral equations (2.12) with the property (2.13) is as follows: We first “fold” the driving semimartingale  $U(\cdot)$  to get its SKOROKHOD reflection  $S(\cdot)$  as in (2.2) and then, starting from the point  $\mathbf{x} = (x_1, x_2)' \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$  with  $x_i = f_i(\mathbf{x}) S(0)$ ,  $i = 1, 2$  and up until the time  $\tau(0)$  of Theorem 2.1(i), we run the planar process  $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$  according to the integral equation

$$X_i(\cdot) = x_i + \int_0^\cdot f_i(X(t)) dS(t), \quad \text{for } i = 1, 2 \quad (3.1)$$

on  $[0, \tau(0))$ . This is the equation to which (2.12) reduces on the interval  $[0, \tau(0))$ . By the definition of the function  $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2)'$  of (2.8), the motion of the two-dimensional process  $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$  during the time-interval  $[0, \tau(0))$  is along the ray that connects the origin  $\mathbf{0}$  to the starting point  $\mathbf{x}$ .

Here is an argument for this claim, which proceeds by applying ITÔ's rule to the stochastic integral equation (2.12) in conjunction with (2.13): Given  $\varepsilon \in (0, \|\mathbf{x}\|)$ , let us define the stopping time  $\sigma_\varepsilon := \inf\{t > 0 : S(t) = \|X(t)\| \leq \varepsilon\}$ . Let us recall that the local time  $L^S(\cdot)$  is flat off the random set  $\mathfrak{Z}$  in (2.3); thus (2.12) reduces to (3.1) on  $[0, \sigma_\varepsilon]$ . Applying ITÔ's rule, we observe

$$\begin{aligned} \frac{X_i(t \wedge \sigma_\varepsilon)}{\|X(t \wedge \sigma_\varepsilon)\|} &= \frac{x_i}{\|\mathbf{x}\|} + \int_0^{t \wedge \sigma_\varepsilon} \frac{dX_i(s)}{\|X(s)\|} - \int_0^{t \wedge \sigma_\varepsilon} \frac{X_i(s)}{\|X(s)\|^2} d\|X(s)\| \\ &\quad + \int_0^{t \wedge \sigma_\varepsilon} \frac{X_i(s)}{\|X(s)\|^3} d\langle \|X\| \rangle(s) - \int_0^{t \wedge \sigma_\varepsilon} \frac{1}{\|X(s)\|^2} d\langle X_i, \|X\| \rangle(s) \end{aligned} \quad (3.2)$$

for  $t \geq 0$  and  $i = 1, 2$ . Because of the definition of  $\mathbf{f}(\cdot)$ , (3.1) and (2.13), we obtain

$$\begin{aligned} X_i(t) &= \|X(t)\| \mathbf{f}_i(X(t)), \quad dX_i(t) = \mathbf{f}_i(X(t)) dS(t) = \frac{X_i(t)}{\|X(t)\|} d\|X(t)\|, \\ \langle X_i, \|X\| \rangle(t) &= \int_0^t \mathbf{f}_i(X(s)) d\langle \|X\| \rangle(s) = \int_0^t \frac{X_i(s)}{\|X(s)\|} d\langle \|X\| \rangle(s) \end{aligned}$$

on  $[0, \sigma_\varepsilon]$ , for  $i = 1, 2$ . Substituting these relations into (3.2), we deduce

$$\mathbf{f}_i(X(t)) = \frac{X_i(t)}{\|X(t)\|} = \frac{x_i}{\|\mathbf{x}\|} = \mathbf{f}_i(\mathbf{x}) \quad \text{on } [0, \sigma_\varepsilon], \quad i = 1, 2.$$

Since  $\varepsilon > 0$  is arbitrary, this concludes the proof of the above claim, in accordance with (2.14).

Now, every time the planar process  $X(\cdot)$  visits the origin, the direction of the next ray for its  $S(\cdot)$ -governed motion is instantaneously chosen at random according to the probability distribution  $\mu$ , the “spinning measure” of the process  $X(\cdot)$ , in a manner described in more detail later. If the origin is visited infinitely often during a time-interval of finite length, it is not surprising that this random choice should lead to the accumulation of local time at the origin, as indicated in the equations (2.12). It follows from (2.17) that the set of times spent by  $X(\cdot)$  at the origin has zero Lebesgue measure. The process continues to move then along the newly chosen ray, its motion governed by the stochastic integral equations of (3.1) just described, as long as it stays away from the origin. The path  $t \mapsto X(t)$  is, with probability one, continuous in the topology induced by the tree-metric (French railway metric) on the plane, namely

$$\varrho(x, y) := (r_1 + r_2) \mathbf{1}_{\{\theta_1 \neq \theta_2\}} + |r_1 - r_2| \mathbf{1}_{\{\theta_1 = \theta_2\}}, \quad x = (r_1, \theta_1), \quad y = (r_2, \theta_2). \quad (3.3)$$

The reader may find it useful at this juncture to think of a roundhouse at the origin, of the spokes of a bicycle wheel – or of the Aeolian winds of Homeric lore, that blow the raft of Odysseus in all directions at once.

### 3.1 Spider Semimartingales

Suppose that the measure  $\mu$  charges only a finite number  $m$  of points on the unit circumference. We can think then of the planar process  $X(\cdot)$  in Theorem 2.1 as a *Spider Semimartingale*, whose radial part  $\|X(\cdot)\| = S(\cdot)$  is the SKOROKHOD reflection of the driver  $U(\cdot)$  according to (2.13).

When the driving semimartingale  $U(\cdot)$  is Brownian motion, the process  $X(\cdot)$  of Theorem 2.1 becomes the original WALSH *Brownian Motion* (as constructed, for instance, in BARLOW, PITMAN & YOR (1989)) with roundhouse singularity in a multipole field; this will be shown in Proposition 7.2 below. When  $m = 2$  and  $\nu(\{0\}) = \alpha \in (0, 1)$ ,  $\nu(\{\pi\}) = 1 - \alpha$ , this construction recovers the familiar *Skew Brownian Motion*, introduced in ITÔ & MCKEAN (1963) and studied by WALSH (1978) and by HARRISON & SHEPP (1981).

### 3.2 Generalized Skew-TANAKA and HARRISON-SHEPP Equations

In the context of Theorem 2.1 (in particular, with the property (2.13)), the equations of (2.12) can be cast in equivalent forms, now driven by the original semimartingale  $U(\cdot)$ , as follows:

$$X_i(\cdot) = x_i + \int_0^\cdot \mathbf{f}_i(X(t)) dU(t) + \gamma_i L^{\|X\|}(\cdot), \quad i = 1, 2, \quad (3.4)$$

$$X_i(\cdot) = x_i + \int_0^\cdot \mathbf{f}_i(X(t)) dU(t) + \left(1 - \frac{\alpha_i^{(-)}}{\alpha_i^{(+)}}\right) L^{X_i}(\cdot), \quad i = 1, 2 \quad (3.5)$$

(the latter when  $\alpha_i^{(+)} > 0$ ). This last system (3.5) is a two-dimensional version of the *skew-TANAKA equation* studied by ICHIBA & KARATZAS (2014).

The system of equations (3.4), on the other hand, can be thought of as a planar semimartingale version of the HARRISON & SHEPP (1981) equation for the skew Brownian motion. For two fixed real constants  $\gamma_1, \gamma_2$ , and a folded driver  $S(\cdot)$  that satisfies the condition

$$\mathbb{P}(L^S(\infty) > 0) > 0 \quad (3.6)$$

(e.g., reflecting Brownian motion), we have the following necessary and sufficient condition (3.7), for the solvability of the system (3.4) subject to the condition (2.13). The requirement (3.7) is a two-dimensional analogue of the condition in HARRISON & SHEPP (1981) for the solvability of the stochastic equation that characterizes the skew Brownian motion. The proof of Theorem 3.1 right below, is given in section 5.

**Theorem 3.1. A Generalized HARRISON-SHEPP System of Equations:** *Consider a continuous semimartingale  $U(\cdot)$  along with its SKOROKHOD reflection  $S(\cdot)$  as in section 2.1, two real numbers  $\gamma_1, \gamma_2$ , and a vector  $\mathbf{x} := (x_1, x_2)' \in \mathbb{R}^2$  with  $x_i = \mathbf{f}_i(\mathbf{x}) S(0)$ ,  $i = 1, 2$ .*

(i) *Suppose that  $\gamma_1, \gamma_2$  satisfy the condition*

$$\gamma_1^2 + \gamma_2^2 \leq 1. \quad (3.7)$$

*There exists then a continuous planar semimartingale  $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$  that satisfies the system (3.4) and the condition (2.13).*

(ii) *Conversely, suppose that (3.6) holds, and that there exists a continuous planar semimartingale  $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$  that satisfies the system (3.4) and the condition (2.13). Then (3.7) is satisfied by  $\gamma_1, \gamma_2$ .*

### 3.3 Open Questions

- It would be of considerable interest to extend the methodology of this paper to a situation with an entire family  $U(\cdot; \mathbf{z})$ ,  $\mathbf{z} \in \mathfrak{S}$  of semimartingales so that, when the point  $\mathbf{z}$  is selected on the unit circumference by the spinning measure  $\boldsymbol{\mu}$ , the motion along the corresponding ray is according to the SKOROKHOD reflection  $S(\cdot; \mathbf{z})$  of this semimartingale  $U(\cdot; \mathbf{z})$ . Some results in this vein are obtained in section 8, in the context of the diffusion case and by the method of scale function and time-change.
- What are the descriptive statistics of the WALSH semimartingale? For example, what is the area of the convex hull of its path  $\{X(s), 0 \leq s \leq T\}$  for some time  $T$ ?

## 4 A FREIDLIN-SHEU-type Formula

Let us consider now a twice continuously differentiable function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ . If  $X(\cdot)$  is a continuous planar semimartingale that satisfies the system of equations (2.12) with the property (2.13), an application of ITÔ's rule with the notation of (2.8) gives

$$\begin{aligned}
g(X(T)) &= g(x) + \int_0^T \left( \sum_{i=1}^2 D_i g(X(t)) \mathbf{f}_i(X(t)) \right) dS(t) + \sum_{i=1}^2 D_i g(\mathbf{0}) \gamma_i \cdot L^S(T) \\
&\quad + \frac{1}{2} \int_0^T \left( \sum_{i=1}^2 \sum_{j=1}^2 D_{ij}^2 g(X(t)) \mathbf{f}_i(X(t)) \mathbf{f}_j(X(t)) \right) d\langle S \rangle(t), \quad 0 \leq T < \infty.
\end{aligned}$$

We define now the functions

$$G'(x) := \sum_{i=1}^2 D_i g(x) \mathbf{f}_i(x), \quad G''(x) := \sum_{i=1}^2 \sum_{j=1}^2 D_{ij}^2 g(x) \mathbf{f}_i(x) \mathbf{f}_j(x) \quad (4.1)$$

on the punctured plane  $E = \mathbb{R}^2 \setminus \{\mathbf{0}\}$ , and consider them as the first and second derivatives, respectively, of the function  $g(\cdot)$  in its radial argument  $r = \sqrt{x_1^2 + x_2^2}$ . With this notation and that of (2.10), (2.11), the above decomposition can be written in the FREIDLIN-SHEU (2000) form

$$g(X(\cdot)) = g(x) + \int_0^\cdot \mathbf{1}_{\{X(t) \neq \mathbf{0}\}} \left( G'(X(t)) dS(t) + \frac{1}{2} G''(X(t)) d\langle S \rangle(t) \right) + \sum_{i=1}^2 \gamma_i D_i g(\mathbf{0}) \cdot L^S(\cdot). \quad (4.2)$$

We also note that the constant  $\sum_{i=1}^2 \gamma_i D_i g(\mathbf{0})$  which multiplies the local time term in (4.2), can be cast as

$$\sum_{i=1}^2 \gamma_i D_i g(\mathbf{0}) = \int_0^{2\pi} h(\theta) \nu(d\theta), \quad \text{the integral of } h(\theta) := \lim_{\|x\| \downarrow 0} G'(x) \Big|_{\arg(x) = \theta} \quad (4.3)$$

with respect to the spinning measure expressed here in polar coordinates, as in HAJRI & TOUHAMI (2014).

#### 4.1 A Generalization of the Change-of-Variable Formula (4.2)

Let us try to refine the above considerations. It is clear that, along the paths of the process  $X(\cdot)$  constructed in Theorem 2.1, only derivatives of the form indicated in (4.1) – i.e., radial – appear in the FREIDLIN-SHEU-like formula (4.2). This suggests that the smoothness assumption in (4.2) can be relaxed, as follows.

**Definition 4.1.** We consider the class  $\mathfrak{D}$  of BOREL-measurable functions  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  with the properties:

- (i) they are continuous in the topology induced by the tree-metric (3.3) on the plane;
- (ii) for every  $\theta \in [0, 2\pi)$ , the function  $r \mapsto g_\theta(r) := g(r, \theta)$  is twice continuously differentiable on  $(0, \infty)$  and has finite first and second right-derivatives at the origin;
- (iii) the resulting functions  $(r, \theta) \mapsto g'_\theta(r)$  and  $(r, \theta) \mapsto g''_\theta(r)$  are BOREL measurable; and
- (iv)  $\sup_{\substack{0 < r < K \\ \theta \in [0, 2\pi)}} (|g'_\theta(r)| + |g''_\theta(r)|) < \infty$  holds for all  $K \in (0, \infty)$ .

Here we consider BOREL sets with respect to the Euclidean topology. We introduce also the subclasses

$$\mathfrak{D}^\mu := \left\{ g \in \mathfrak{D} : \int_0^{2\pi} g'_\theta(0+) \nu(d\theta) = 0 \right\}, \quad \mathfrak{D}_+^\mu := \left\{ g \in \mathfrak{D} : \int_0^{2\pi} g'_\theta(0+) \nu(d\theta) \geq 0 \right\}. \quad (4.4)$$

**Definition 4.2.** For every given function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  in  $\mathfrak{D}$  we set by analogy with (4.1):  $G'(x) := g'_\theta(r)$  and  $G''(x) := g''_\theta(r)$  for  $x = (r, \theta)$  with  $r > 0$ .

With this notation in place, we can formulate our second major result.

**Theorem 4.1. A Generalized FREIDLIN-SHEU Formula:** *With the above notation, every continuous semimartingale  $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$  which solves the system of equations (2.12) and satisfies the properties (2.13) and (2.18), also satisfies for every  $g \in \mathfrak{D}$  the generalized FREIDLIN-SHEU identity*

$$\begin{aligned} g(X(\cdot)) &= g(x) + \int_0^\cdot \mathbf{1}_{\{X(t) \neq 0\}} \left( G'(X(t)) dS(t) + \frac{1}{2} G''(X(t)) d\langle S \rangle(t) \right) + \left( \int_0^{2\pi} g'_\theta(0+) \nu(d\theta) \right) L^S(\cdot) \\ &= g(x) + \int_0^\cdot \mathbf{1}_{\{X(t) \neq 0\}} \left( G'(X(t)) dU(t) + \frac{1}{2} G''(X(t)) d\langle U \rangle(t) \right) + \left( \int_0^{2\pi} g'_\theta(0+) \nu(d\theta) \right) L^S(\cdot). \end{aligned} \quad (4.5)$$

In particular, the continuous semimartingale  $X(\cdot)$  of Theorem 2.1 satisfies (4.5).

## 4.2 Slope-Averaging Martingales

For any given bounded, measurable  $\varphi : [0, 2\pi) \rightarrow \mathbb{R}$ , let us define the functions

$$h_{(\varphi)}(x) := (\varphi(\arg(x)) - \mathbb{E}[\varphi(\arg(\xi_1))]) \cdot \mathbf{1}_{\{x \neq 0\}}, \quad g_{(\varphi)}(x) := \|x\| \cdot h_{(\varphi)}(x) \quad (4.6)$$

for  $x \in \mathbb{R}^2$ , where  $\xi_1$  is an  $\mathfrak{S}$ -valued random variable with distribution  $\mu$  as in (5.1). Such functions were first introduced by BARLOW, PITMAN & YOR (1989), in their study of the WALSH Brownian motion. Using polar coördinates, we observe that  $g_{(\varphi)}(x) \equiv g_{(\varphi)}(r, \theta)$  belongs to the class  $\mathfrak{D}$  and satisfies

$$G'_{(\varphi)}(x) \equiv (g_{(\varphi)})'_\theta(r) = h_{(\varphi)}(r, \theta), \quad G''_{(\varphi)}(x) \equiv (g_{(\varphi)})''_\theta(r) = 0, \quad \int_0^{2\pi} h_{(\varphi)}(r, \theta) \nu(d\theta) = 0.$$

Here  $'$  denotes differentiation with respect to  $r \in (0, \infty)$ .

Direct application of Theorem 4.1 gives the following result.

**Proposition 4.1.** *Assume that  $U(\cdot)$  in (2.1) is a continuous local martingale, and construct its SKOROKHOD reflection  $S(\cdot)$  as in (2.2). Consider any continuous semimartingale  $X(\cdot) := (X_1(\cdot), X_2(\cdot))'$  which satisfies the system of equations (2.12), along with the properties (2.13) and (2.18).*

(i) *For any function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  in the class  $\mathfrak{D}^\mu$  of (4.4), the process*

$$g(X(\cdot)) - g(X(0)) - \frac{1}{2} \int_0^\cdot G''(X(t)) \mathbf{1}_{\{X(t) \neq 0\}} d\langle U \rangle(t)$$

*is a continuous local martingale with quadratic variation  $\int_0^\cdot (G'(X(t)))^2 \mathbf{1}_{\{X(t) \neq 0\}} d\langle U \rangle(t)$ .*

(ii) *For any given bounded, measurable function  $\varphi : [0, 2\pi) \rightarrow \mathbb{R}$  and with the notation of (4.6), the process below is a continuous local martingale:*

$$g_{(\varphi)}(X(\cdot)) = \|X(\cdot)\| h_{(\varphi)}(X(\cdot)) = g_{(\varphi)}(x) + \int_0^\cdot h_{(\varphi)}(X(t)) dU(t).$$

## 5 The Proofs of Theorems 2.1, 4.1 and 3.1

The way we construct a process  $X(\cdot)$  which satisfies the equation (2.12) is via “folding and unfolding of semimartingales”, with additional randomness coming from a sequence  $\xi_1, \xi_2, \dots$  of  $\mathfrak{S}$ -valued, I.I.D. random variables. These have common probability distribution  $\mu$  on  $\mathfrak{S}$ , such that the components of the random vector  $\xi_1 := (\xi_{1,1}, \xi_{1,2})'$  have expectations that are matched with the parameter vector  $(\alpha_1^{(+)}, \alpha_1^{(-)}, \alpha_2^{(+)}, \alpha_2^{(-)}) \in [0, 1]^4$  in (2.10), (2.12) as

$$\mathbb{E}(\xi_{1,i}^\pm) = \alpha_i^{(\pm)}, \quad \mathbb{E}(\xi_{1,i}) = \alpha_i^{(+)} - \alpha_i^{(-)} = \gamma_i, \quad \mathbb{E}(|\xi_{1,i}|) = \alpha_i^{(+)} + \alpha_i^{(-)}; \quad i = 1, 2. \quad (5.1)$$



**Proof of Theorem 2.1:** For simplicity, we consider the case  $x_1 = x_2 = 0$  first. Following PROKAJ (2009) and ICHIBA & KARATZAS (2014), we enlarge the original probability space by means of the above sequence  $\{\xi_k\}_{k \in \mathbb{N}}$  of  $\mathfrak{S}$ -valued, I.I.D. random variables. These are independent of the  $\sigma$ -algebra  $\tilde{\mathcal{F}}(\infty) := \bigvee_{0 \leq t < \infty} \tilde{\mathcal{F}}(t)$  and have expectation  $\mathbb{E}(\xi_1) = \gamma$  as in (5.1), (2.11).

- Let us decompose the nonnegative half-line into the zero set  $\mathfrak{Z}$  of  $S(\cdot)$  as in (2.3) on the one hand, and the countable collection  $\{\mathcal{C}_k\}_{k \in \mathbb{N}}$  of open disjoint components of  $[0, \infty) \setminus \mathfrak{Z}$  on the other. Each of these components represents an excursion interval away from the origin for the SKOROKHOD reflection process  $S(\cdot)$  in (2.2). Here we enumerate these countably-many excursion intervals  $\{\mathcal{C}_k\}_{k \in \mathbb{N}}$  in a measurable manner, so that  $\{t \in \mathcal{C}_k\} \in \tilde{\mathcal{F}}(\infty)$  holds for all  $t \geq 0$ ,  $k \in \mathbb{N}$ . For notational simplicity, we declare also  $\mathcal{C}_0 := \mathfrak{Z}$ ,  $\xi_0 := \mathbf{0}$ . We shall denote

$$Z(t) := \sum_{k \in \mathbb{N}_0} \xi_k \cdot \mathbf{1}_{\mathcal{C}_k}(t), \quad X(t) := Z(t)S(t), \quad \mathcal{F}^Z(t) := \sigma(Z(s), 0 \leq s \leq t) \quad (5.2)$$

for  $0 \leq t < \infty$  and introduce the enlarged filtration  $\mathbb{F} := \{\mathcal{F}(t)\}_{0 \leq t < \infty}$  via  $\mathcal{F}(t) := \tilde{\mathcal{F}}(t) \vee \mathcal{F}^Z(t)$ .

This procedure corresponds exactly to the program outlined by J.B. WALSH in the appendix to his 1978 paper, as follows: “The idea is to take each excursion of (reflecting Brownian motion) and, instead of giving a random sign, to assign it a random variable with a given distribution in  $[0, 2\pi)$ , and to do so independently for each excursion”. Of course the process  $S(\cdot)$  is one-dimensional, while  $Z(\cdot) = (Z_1(\cdot), Z_2(\cdot))'$  and  $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$  are two-dimensional processes with  $f(X(\cdot)) = (f_1(X(\cdot)), f_2(X(\cdot)))' = Z(\cdot)$ :

$$f_i(X(\cdot)) = Z_i(\cdot), \quad i = 1, 2 \quad (5.3)$$

with the functions  $f_i(\cdot)$  as defined in (2.8). In particular, the zero set of (2.3) is

$$\mathfrak{Z} = \{t \geq 0 : S(t) = 0\} = \{t \geq 0 : Z(t) = \mathbf{0}\} = \{t \geq 0 : X(t) = \mathbf{0}\}. \quad (5.4)$$

We can also think of the vector process  $X(\cdot)$  of (5.2) as expressed in its polar coordinates

$$S(t) = \sqrt{X_1^2(t) + X_2^2(t)} \quad \text{and} \quad \Theta(t) = \arg(Z(t)) = \sum_{k \in \mathbb{N}} \arg(\xi_k) \cdot \mathbf{1}_{\mathcal{C}_k}(t) \quad (5.5)$$

from (5.2). We shall see presently that this process  $X(\cdot)$  satisfies the system of equations (2.12).

- We claim that, because of independence and of the way the probability space was enlarged, *both processes  $U(\cdot)$  and  $S(\cdot)$  are continuous  $\mathbb{F}$ -semimartingales*. This claim can be established as in the proof of Proposition 2 in PROKAJ (2009); see also Proposition 3.1 in ICHIBA & KARATZAS (2014).

- In order to describe the dynamics of the process  $X(\cdot)$  defined in (5.2), we approximate the process  $Z(\cdot)$  also defined there by a family of processes  $Z^\varepsilon(\cdot)$  with finite first variation over compact intervals indexed by  $\varepsilon \in (0, 1)$ , as follows. We define the sequence of stopping times  $\tau_0^\varepsilon := \inf \{t \geq 0 : \|X(t)\| = 0\}$  and

$$\tau_{2\ell+1}^\varepsilon := \inf \{t > \tau_{2\ell}^\varepsilon : \|X(t)\| \geq \varepsilon\}, \quad \tau_{2\ell+2}^\varepsilon := \inf \{t > \tau_{2\ell+1}^\varepsilon : \|X(t)\| = 0\}; \quad \ell \in \mathbb{N}_0 \quad (5.6)$$

recursively. We also introduce a piecewise-constant process  $Z^\varepsilon(\cdot) := (Z_1^\varepsilon(\cdot), Z_2^\varepsilon(\cdot))'$  with

$$Z^\varepsilon(t) := \sum_{\ell \in \mathbb{N}_0} Z(t) \mathbf{1}_{[\tau_{2\ell+1}^\varepsilon, \tau_{2\ell+2}^\varepsilon)}(t) = \sum_{(k, \ell) \in \mathbb{N}_0^2} \xi_k \mathbf{1}_{\mathcal{C}_k \cap [\tau_{2\ell+1}^\varepsilon, \tau_{2\ell+2}^\varepsilon)}(t), \quad 0 \leq t < \infty, \quad (5.7)$$

i.e., constant on each of the “downcrossing intervals”  $[\tau_{2\ell+1}^\varepsilon, \tau_{2\ell+2}^\varepsilon)$ . For this process, the product rule gives

$$X^\varepsilon(T) := Z^\varepsilon(T)S(T) = \int_0^T Z^\varepsilon(t) dS(t) + \int_0^T S(t) dZ^\varepsilon(t), \quad 0 \leq T < \infty. \quad (5.8)$$

Passing to the limit as  $\varepsilon \downarrow 0$  and using (5.1)-(5.3), as well as the characterization of the local time  $L^S(\cdot)$  of the semimartingale  $S(\cdot)$  in terms of the number of its downcrossings, we obtain the decomposition

$$X(T) = Z(T)S(T) = \int_0^T Z(t) dS(t) + \mathbb{E}[\xi_1] L^S(T) = \int_0^T f(X(t)) dS(t) + \gamma L^S(T) \quad (5.9)$$

in the notation of (2.11). Indeed, the second term on the right-hand side of (5.8) can be estimated by the strong law of large numbers and Theorem VI.1.10 in REVUZ & YOR (1999): namely, we have

$$\begin{aligned} \int_0^T S(t) dZ^\varepsilon(t) &= \sum_{\{\ell: \tau_{2\ell+1}^\varepsilon < T\}} S(\tau_{2\ell+1}^\varepsilon) Z^\varepsilon(\tau_{2\ell+1}^\varepsilon) = \varepsilon \sum_{j=1}^{N(T,\varepsilon)} \xi_{\ell_j} + O(\varepsilon) \\ &= \varepsilon N(T,\varepsilon) \cdot \frac{1}{N(T,\varepsilon)} \sum_{j=1}^{N(T,\varepsilon)} \xi_{\ell_j} + O(\varepsilon) \xrightarrow{\varepsilon \downarrow 0} L^S(T) \cdot \mathbb{E}[\xi_1] \end{aligned} \quad (5.10)$$

in probability. Here  $\{\xi_{\ell_j}\}_{j=1}^{N(T,\varepsilon)}$  is an enumeration of  $Z^\varepsilon(\tau_{2\ell+1}^\varepsilon)$ , and  $N(T,\varepsilon) := \#\{\ell \in \mathbb{N} : \tau_{2\ell}^\varepsilon < T\}$  the number of downcrossings of the interval  $(0, \varepsilon)$  that the process  $S(\cdot)$  has completed during  $[0, T]$ . We deduce from (5.9), in particular, that the process  $X(\cdot)$  is a continuous planar  $\mathbb{F}$ -semimartingale.

By analogy with (5.9), we can approximate the process  $|Z_i(\cdot)|$  by  $|Z_i^\varepsilon(\cdot)|$ , the absolute value of each of the components  $Z_i^\varepsilon(\cdot)$  of the piecewise-constant process in (5.7); passing to the limit as  $\varepsilon \downarrow 0$ , we obtain

$$|X_i(T)| = |Z_i(T)|S(T) = \int_0^T |Z_i(t)| dS(t) + \mathbb{E}(|\xi_{1,i}|) L^S(T), \quad 0 \leq T < \infty \quad (5.11)$$

for  $i = 1, 2$ . We appeal now to Exercise VI (1.16) 3° of REVUZ & YOR (1999); recalling the form of  $S(\cdot)$  in (2.2) along with (5.4) we deduce that, with the normalization of (2.5), the continuous, nonnegative semimartingale  $|X_i(\cdot)|$ ,  $i = 1, 2$  with the decomposition (5.11) has local time at the origin

$$L^{|X_i|}(\cdot) = \int_0^\cdot \mathbf{1}_{\{X_i(t)=0\}} [|Z_i(t)| dS(t) + (\alpha_i^{(+)} + \alpha_i^{(-)}) dL^S(t)] = (\alpha_i^{(+)} + \alpha_i^{(-)}) L^S(\cdot). \quad (5.12)$$

• At this point, we need to identify the local times  $L^{X_i}(\cdot)$  of each component  $X_i(\cdot)$  in terms of the local time  $L^S(\cdot)$ . Since  $X_i(\cdot) = Z_i(\cdot)S(\cdot)$  is a continuous semimartingale for  $i = 1, 2$ , we recall the decomposition (5.9) and properties of semimartingale local time and obtain the string of identities

$$\begin{aligned} 2L^{X_i}(\cdot) - L^{|X_i|}(\cdot) &= \int_0^\cdot \mathbf{1}_{\{X_i(t)=0\}} dX_i(t) = \int_0^\cdot \mathbf{1}_{\{X_i(t)=0\}} [Z_i(t) dS(t) + \mathbb{E}(\xi_{1,i}) dL^S(t)] \\ &= \mathbb{E}(\xi_{1,i}) L^S(\cdot) = (\alpha_i^{(+)} - \alpha_i^{(-)}) L^S(\cdot) \end{aligned} \quad (5.13)$$

(cf. subsection 2.1 in ICHIBA, KARATZAS & PROKAJ (2013)). Thus, combining with (5.12), we deduce

$$2L^{X_i}(\cdot) = (\mathbb{E}(|\xi_{1,i}|) + \mathbb{E}(\xi_{1,i})) L^S(\cdot) = 2\mathbb{E}(\xi_{1,i}^+) L^S(\cdot) = 2\alpha_i^{(+)} L^S(\cdot), \quad i = 1, 2, \quad (5.14)$$

i.e., property (2.16). The equations (2.12) and (2.13), (2.17) follow now from (2.4), (2.10), (5.5) and (5.9).

• The property (2.18) can be shown by an approximation similar in spirit and manner to that just carried out in the proof of (2.16). We take now throughout the “thinned” sequence  $\xi_k^A := \mathbf{1}_A(\arg(\xi_k))$ ,  $k \in \mathbb{N}_0$  in place of  $\xi_k$ ,  $k \in \mathbb{N}_0$ ; and in lieu of  $Z(\cdot)$  and  $S(\cdot)$  in (5.2), respectively, the processes

$$Z^{(A)}(\cdot) := \sum_{k \in \mathbb{N}_0} \xi_k^A \cdot \mathbf{1}_{C_k}(\cdot) \quad \text{and} \quad R^{(A)}(\cdot) = \|X(\cdot)\| \cdot \mathbf{1}_A(\arg(X(\cdot))) = S(\cdot) Z^{(A)}(\cdot).$$

• When the initial value  $x = (x_1, x_2)'$  is not the origin, we define

$$Z(0) := f(X(0)) = f(x) \quad \text{and} \quad X(t) := Z(0)S(t), \quad \text{for } 0 \leq t < \tau(0),$$

i.e., until the process  $X(\cdot)$  first attains the origin, very much in accordance with (3.1). Here  $\tau(0)$  is defined as in Theorem 2.1(i). The so-constructed process  $X(\cdot)$  satisfies the stochastic differential equation (3.1), to which (2.12) reduces on the interval  $[0, \tau(0))$  as in the discussion at the start of section 3. On the interval  $[\tau(0), \infty)$  we use the recipe (5.2) above, to construct  $X(\cdot)$  starting from the origin.

With these considerations we obtain  $\{f(X(s)) = f(x), s < \tau(0)\} = \{s < \tau(0)\}$ , mod.  $\mathbb{P}$ , and hence we verify (2.14). Moreover, for every  $(s, t) \in (0, \infty)^2$ , there exists by construction an  $\tilde{\mathcal{F}}(\infty)$ -measurable random index  $\kappa_0(s, t) : \Omega \rightarrow \mathbb{N}$  such that we have, either  $\tau(s) + t \in \mathcal{C}_{\kappa_0(s, t)}$ , or  $\tau(s) + t \in \mathfrak{Z}$  on  $\{\tau(s) < \infty\}$ . If  $\tau(s) + t \in \mathfrak{Z}$  and  $\tau(s) < \infty$ , then  $f(X(\tau(s) + t)) = \mathbf{0}$ . By (2.4) and the construction of  $X(\cdot)$  we obtain  $\mathbb{P}(f(X(\tau(s) + t)) = \mathbf{0}) = \mathbb{P}(S(\tau(s) + t) = 0) = 0$  for a.e.  $t \in (0, \infty)$ . Therefore,

$$\{f(X(\tau(s) + t)) \in B, \tau(s) < \infty\} = \left\{ \sum_{k \in \mathbb{N}_0} \xi_k \mathbf{1}_{\mathcal{C}_k}(\tau(s) + t) \in B, \tau(s) < \infty \right\} = \{\xi_{\kappa_0(s, t)} \in B, \tau(s) < \infty\}$$

holds mod.  $\mathbb{P}$  for every  $B \in \mathcal{B}(\mathfrak{S})$  and almost every  $t \in (0, \infty)$ . We conclude that (2.15) holds, namely

$$\begin{aligned} \mathbb{P}(f(X(\tau(s) + t)) \in B \mid \mathcal{F}^X(\tau(s))) &= \mathbb{P}(\xi_{\kappa_0(s, t)} \in B \mid \mathcal{F}^X(\tau(s))) \\ &= \mathbb{E} \left[ \mathbb{P}[\xi_{\kappa_0(s, t)} \in B \mid \tilde{\mathcal{F}}(\infty) \vee \mathcal{F}^Z(\tau(s))] \mid \mathcal{F}^X(\tau(s)) \right] = \mathbb{E}[\mathbb{P}(\xi_1 \in B) \mid \mathcal{F}^X(\tau(s))] = \mu(B), \end{aligned}$$

for every  $s \in (0, \infty)$ ,  $B \in \mathcal{B}(\mathfrak{S})$  and almost every  $t \in (0, \infty)$ . We have used here the definitions of  $\mathcal{F}^X(\cdot) \subseteq \tilde{\mathcal{F}}(\cdot) \vee \mathcal{F}^Z(\cdot)$  and  $Z(\cdot)$  in (5.2), the  $\tilde{\mathcal{F}}(\infty)$ -measurability of the stopping time  $\tau(s)$  and of the random index  $\kappa_0(s, t)$ , and the independence between  $\tilde{\mathcal{F}}(\infty)$  and the sequence  $\{\xi_k\}_{k \in \mathbb{N}}$  of I.I.D. random variables. This completes the proof of Theorem 2.1.  $\square$

**Proof of Theorem 4.1:** Let us fix a function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  in the class  $\mathfrak{D}$  as in the statement of the theorem, and recall the notation established in Definitions 4.1, 4.2. Consider also a continuous planar semimartingale  $X(\cdot)$  satisfying the equations of (2.12) along with the properties (2.13) and (2.18). With  $\{\tau_k^\varepsilon\}_{k \in \mathbb{N}_0}$  defined as in (5.6), and with  $\tau_{-1}^\varepsilon \equiv 0$  and  $\mathbb{N}_{-1} := \mathbb{N}_0 \cup \{-1\}$ , the value  $g(X(T))$  is decomposed into

$$g(X(T)) = g(x) + \sum_{\ell \in \mathbb{N}_{-1}} (g(X(T \wedge \tau_{2\ell+2}^\varepsilon)) - g(X(T \wedge \tau_{2\ell+1}^\varepsilon))) + \sum_{\ell \in \mathbb{N}_0} (g(X(T \wedge \tau_{2\ell+1}^\varepsilon)) - g(X(T \wedge \tau_{2\ell}^\varepsilon))). \quad (5.15)$$

• We recall from the discussion at the beginning of section 3, that the process  $X(\cdot)$  moves along the ray that connects  $\mathbf{0}$  to the starting point  $x \neq \mathbf{0}$ , during the time-interval  $[0, \tau(0)) = [\tau_{-1}^\varepsilon, \tau_0^\varepsilon)$ . In a similar manner, the processes  $f_i(X(\cdot))$  are constant on every interval  $[\tau_{2\ell+1}^\varepsilon, \tau_{2\ell+2}^\varepsilon)$  for  $\ell \in \mathbb{N}_0$ ,  $i = 1, 2$ .

The first summation in (5.15) can thus be rewritten as

$$\begin{aligned} \sum_{\ell \in \mathbb{N}_{-1}} (g(X(T \wedge \tau_{2\ell+2}^\varepsilon)) - g(X(T \wedge \tau_{2\ell+1}^\varepsilon))) &= \sum_{\ell \in \mathbb{N}_{-1}} \left( g_\theta(S(T \wedge \tau_{2\ell+2}^\varepsilon)) - g_\theta(S(T \wedge \tau_{2\ell+1}^\varepsilon)) \right) \Big|_{\theta = \Theta(T \wedge \tau_{2\ell+1}^\varepsilon)} \\ &= \sum_{\ell \in \mathbb{N}_{-1}} \int_{T \wedge \tau_{2\ell+1}^\varepsilon}^{T \wedge \tau_{2\ell+2}^\varepsilon} \left( g'_\theta(S(t)) dS(t) + \frac{1}{2} g''_\theta(S(t)) d\langle S \rangle(t) \right) \Big|_{\theta = \Theta(t)} \\ &= \int_0^T \left( \sum_{\ell \in \mathbb{N}_{-1}} \mathbf{1}_{(\tau_{2\ell+1}^\varepsilon, \tau_{2\ell+2}^\varepsilon)}(t) \right) \left( G'(X(t)) dS(t) + \frac{1}{2} G''(X(t)) d\langle S \rangle(t) \right). \end{aligned}$$

We have set here  $\Theta(\cdot) := \arg(X(\cdot))$ , and applied ITÔ's rule (Problem 3.7.3 in KARATZAS & SHREVE (1991)) to the process  $g_\theta(S(\cdot))$ . Letting  $\varepsilon \downarrow 0$ , we obtain in the limit

$$\begin{aligned} \int_0^T \mathbf{1}_{\{X(t) \neq \mathbf{0}\}} \left( G'(X(t)) dS(t) + \frac{1}{2} G''(X(t)) d\langle S \rangle(t) \right) \\ = \int_0^T \mathbf{1}_{\{X(t) \neq \mathbf{0}\}} \left( G'(X(t)) dU(t) + \frac{1}{2} G''(X(t)) d\langle U \rangle(t) \right). \end{aligned} \quad (5.16)$$

- For the second summation in (5.15), we observe  $g(\mathbf{0}) = g_\theta(0)$  by definition and hence

$$\begin{aligned}
& \sum_{\ell \in \mathbb{N}_0} (g(X(T \wedge \tau_{2\ell+1}^\varepsilon)) - g(X(T \wedge \tau_{2\ell}^\varepsilon))) \tag{5.17} \\
&= \sum_{\{\ell: \tau_{2\ell+1}^\varepsilon < T\}} (g_\theta(S(\tau_{2\ell+1}^\varepsilon)) - g_\theta(0)) \Big|_{\theta = \Theta(\tau_{2\ell+1}^\varepsilon)} + O(\varepsilon) = \sum_{\{\ell: \tau_{2\ell+1}^\varepsilon < T\}} (g_\theta(\varepsilon) - g_\theta(0)) \Big|_{\theta = \Theta(\tau_{2\ell+1}^\varepsilon)} + O(\varepsilon) \\
&= \sum_{\{\ell: \tau_{2\ell+1}^\varepsilon < T\}} \left( \varepsilon g'_\theta(0+) + \int_0^\varepsilon (\varepsilon - u) g''_\theta(u) du \right) \Big|_{\theta = \Theta(\tau_{2\ell+1}^\varepsilon)} + O(\varepsilon) \xrightarrow{\varepsilon \downarrow 0} L^S(T) \int_0^{2\pi} g'_\theta(0+) \nu(d\theta)
\end{aligned}$$

in probability. Indeed, by analogy with (5.10) we can verify

$$\left| \sum_{\{\ell: \tau_{2\ell+1}^\varepsilon < T\}} \left( \int_0^\varepsilon (\varepsilon - u) g''_\theta(u) du \right) \Big|_{\theta = \Theta(\tau_{2\ell+1}^\varepsilon)} \right| \leq c \sum_{\{\ell: \tau_{2\ell+1}^\varepsilon < T\}} \varepsilon^2 = c\varepsilon \cdot (\varepsilon N(T, \varepsilon) + O(\varepsilon)) \xrightarrow{\varepsilon \downarrow 0} 0$$

in probability, where  $c := \sup_{\theta \in \text{supp}(\mu)} \max_{0 \leq u \leq 1} (g''_\theta(u) / 2) < +\infty$  by assumption.

- We also check that for every  $A \in \mathcal{B}([0, 2\pi))$  we have, on account of the property (2.18) for the process  $R^A(\cdot) = \|X(\cdot)\| \mathbf{1}_A(\Theta(\cdot))$ , the convergence

$$\begin{aligned}
\sum_{\{\ell: \tau_{2\ell+1}^\varepsilon < T\}} \varepsilon \mathbf{1}_{\{\Theta(\tau_{2\ell+1}^\varepsilon) \in A\}} &= \sum_{\{\ell: \tau_{2\ell+1}^\varepsilon < T\}} S(\tau_{2\ell+1}^\varepsilon) \mathbf{1}_{\{\Theta(\tau_{2\ell+1}^\varepsilon) \in A\}} = \sum_{\{\ell: \tau_{2\ell+1}^\varepsilon < T\}} \|X(\tau_{2\ell+1}^\varepsilon)\| \mathbf{1}_{\{\Theta(\tau_{2\ell+1}^\varepsilon) \in A\}} \\
&= \sum_{\{\ell: \tilde{\tau}_{2\ell+1}^\varepsilon < T\}} R^A(\tilde{\tau}_{2\ell+1}^\varepsilon) = \varepsilon \tilde{N}(T, \varepsilon) + O(\varepsilon) \xrightarrow{\varepsilon \downarrow 0} L^{R^A}(T) = \nu(A) L^S(T)
\end{aligned}$$

in probability. Here we define  $\tilde{\tau}_0^\varepsilon := \inf \{t \geq 0 : R^A(t) = 0\}$ , and recursively

$$\tilde{\tau}_{2\ell+1}^\varepsilon := \inf \{t > \tilde{\tau}_{2\ell}^\varepsilon : R^A(t) \geq \varepsilon\}, \quad \tilde{\tau}_{2\ell+2}^\varepsilon := \inf \{t > \tilde{\tau}_{2\ell+1}^\varepsilon : R^A(t) = 0\}$$

for  $\ell \in \mathbb{N}_0$ , and denote by  $\tilde{N}(\varepsilon, T)$  the number of downcrossings of the interval  $(0, \varepsilon)$  that the process  $R^A(\cdot)$  has completed during the interval  $[0, T)$  (please note that we count here the number of downcrossings corresponding to the rays in the directions in the subset  $A$  of  $[0, 2\pi)$ ). Thus, approximating the function  $\theta \mapsto g'_\theta(0+)$  by indicators  $\theta \mapsto \mathbf{1}_A(\theta)$ ,  $A \in \mathcal{B}([0, 2\pi))$ , we verify the convergence

$$\sum_{\{\ell: \tau_{2\ell+1}^\varepsilon < T\}} \varepsilon g'_{\Theta(\tau_{2\ell+1}^\varepsilon)}(0+) \xrightarrow{\varepsilon \downarrow 0} L^S(T) \int_0^{2\pi} g'_\theta(0+) \nu(d\theta), \quad \text{in probability.} \tag{5.18}$$

- Therefore, the limit of the expression in (5.15) is the sum of the limits of the expressions in (5.16) and (5.17), and we conclude with

$$g(X(T)) = g(x) + \left( \int_0^{2\pi} g'_\theta(0+) \nu(d\theta) \right) L^S(T) + \int_0^T \mathbf{1}_{\{X(t) \neq \mathbf{0}\}} \left( G'(X(t)) dS(t) + \frac{1}{2} G''(X(t)) d\langle S \rangle(t) \right),$$

which establishes (4.5) and proves the first claim in Theorem 4.1; the second follows then directly.  $\square$

**Proof of Theorem 3.1:** (i) Assume  $\gamma_1^2 + \gamma_2^2 \leq 1$ , and consider the vector  $\gamma := (\gamma_1, \gamma_2)' \in \mathbb{R}^2$ . Then we define the probability measure  $\mu := ((1+\beta)/2) \delta_{z_0} + ((1-\beta)/2) \delta_{-z_0}$  on  $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$  with  $\beta := \|\gamma\| \leq 1$  and  $z_0 := \gamma/\beta \in \mathfrak{S}$  provided that  $\beta \neq 0$  (if  $\beta = 0$ , we simply pick up an arbitrary  $z_0 \in \mathfrak{S}$ ), and note

$$\int_{\mathfrak{S}} f(z) \mu(dz) = \int_{\mathfrak{S}} z \mu(dz) = \frac{1+\beta}{2} z_0 + \frac{1-\beta}{2} (-z_0) = \beta z_0 = \gamma.$$

Thus, if we take the process  $S(\cdot)$  in section 2 as the “folded driver” and  $\mu$  as the “spinning measure”, Theorem 2.1 constructs a continuous planar semimartingale  $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$  that satisfies the condition (2.13) and the system of equations (2.12) – thus also the system (3.4).

(ii) Suppose now that (3.6) holds, and that there exists a continuous semimartingale  $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$  which satisfies (2.13) and the system of equations (3.4), thus also of (2.12). For every  $\varepsilon > 0$ , we define  $\tau_{-1}^\varepsilon \equiv 0$  and  $\{\tau_m^\varepsilon\}_{m \in \mathbb{N}_0}$  as in (5.6). Following the idea in the proof of Theorem 4.1, we write

$$X(T) = x + \sum_{\ell \in \mathbb{N}_{-1}} (X(T \wedge \tau_{2\ell+2}^\varepsilon) - X(T \wedge \tau_{2\ell+1}^\varepsilon)) + \sum_{\ell \in \mathbb{N}_0} (X(T \wedge \tau_{2\ell+1}^\varepsilon) - X(T \wedge \tau_{2\ell}^\varepsilon)).$$

Then as  $\varepsilon \downarrow 0$ , on account of (2.12) and in the same manner as in the proof of Theorem 4.1, the first summation in the above expression converges in probability to  $\int_0^T f(X(t)) dU(t)$ . Thus, the second summation converges in probability to  $\gamma L^{\|X\|}(T)$ , thanks to (3.4). This implies the convergence in probability

$$\sum_{\ell \in \mathbb{N}_0} (X(T \wedge \tau_{2\ell+1}^\varepsilon) - X(T \wedge \tau_{2\ell}^\varepsilon)) = \sum_{\ell=0}^{N(T,\varepsilon)-1} \varepsilon f(X(\tau_{2\ell+1}^\varepsilon)) + O(\varepsilon) \xrightarrow{\varepsilon \downarrow 0} \gamma L^{\|X\|}(T).$$

We also have the convergence in probability  $\varepsilon N(T, \varepsilon) \rightarrow L^{\|X\|}(T)$  as  $\varepsilon \downarrow 0$  by Theorem VI.1.10 in REVUZ & YOR (1999), where  $N(T, \varepsilon) := \#\{\ell \in \mathbb{N} : \tau_{2\ell}^\varepsilon < T\}$ . Therefore, we have

$$\frac{1}{N(T, \varepsilon)} \sum_{\ell=0}^{N(T,\varepsilon)-1} f(X(\tau_{2\ell+1}^\varepsilon)) \xrightarrow{\varepsilon \downarrow 0} \gamma \quad \text{in probability, on the event } \{L^{\|X\|}(T) > 0\}.$$

Now  $\|\gamma\| \leq 1$  follows from  $\|f(\cdot)\| \leq 1$ , since we can select a sufficiently large  $T \in (0, \infty)$  such that  $\mathbb{P}(L^{\|X\|}(T) > 0) > 0$  (thanks to (3.6) and (2.13)).  $\square$

## 6 WALSH Diffusions and the Associated Martingale Problems

We cannot expect pathwise uniqueness, therefore neither can we expect strength, to hold for the equations of (2.12) or (3.4). Any such lingering hope is dashed by the realization that, when  $U(\cdot)$  is standard Brownian motion, thus  $S(\cdot)$  a reflecting Brownian motion, the process  $X(\cdot)$  constructed in Theorem 2.1 is the WALSH Brownian motion – a process whose filtration cannot be generated by *any* Brownian motion of *any* dimension. For this result see the celebrated paper by TSIREL'SON (1997), as well as Proposition 7.2 below and MANSUY & YOR (2006), pages 103-116. In light of these observations, it is natural to ask whether the next best thing, that is, *uniqueness in distribution*, might hold for these equations under appropriate conditions. We try in this section to provide some affirmative answers to this question, when the folded driving semimartingale  $S(\cdot)$  is a reflected diffusion; the main results appear in Propositions 6.2 and 6.3.

### 6.1 The Folded Driving Semimartingale as a Reflected Diffusion

Let us start by considering the canonical space  $\Omega_1 := C([0, \infty); [0, \infty))$  of nonnegative, continuous functions on  $[0, \infty)$ . We endow this space with the usual topology of uniform convergence over compact intervals and with the  $\sigma$ -algebra  $\mathcal{F}_1 := \mathcal{B}(\Omega_1)$  of its BOREL sets. We consider also the filtration  $\mathbb{F}_1 := \{\mathcal{F}_1(t)\}_{0 \leq t < \infty}$  generated by its coordinate mapping, i.e.,  $\mathcal{F}_1(t) = \sigma(\omega_1(s), 0 \leq s \leq t)$ .

Given BOREL-measurable coefficients  $\mathbf{b} : [0, \infty) \rightarrow \mathbb{R}$  and  $\sigma : [0, \infty) \rightarrow \mathbb{R} \setminus \{0\}$  and setting  $\mathbf{a}(\cdot) := \sigma^2(\cdot)$ , we define the process

$$K^\psi(\cdot; \omega_1) := \psi(\omega_1(\cdot)) - \psi(\omega_1(0)) - \int_0^\cdot \mathcal{G}\psi(\omega_1(t)) \cdot \mathbf{1}_{\{\omega_1(t) > 0\}} dt, \quad (6.1)$$

where

$$\mathcal{G}\psi(r) := \mathbf{b}(r) \psi'(r) + \frac{1}{2} \mathbf{a}(r) \psi''(r); \quad r \in [0, \infty), \quad \psi \in C_0^2([0, \infty); \mathbb{R}).$$

### 6.1.1 Local Submartingale Problem for a Reflected Diffusion

In the manner of STROOCK & VARADHAN (1971), we formulate the **Local Submartingale Problem associated with the pair  $(\sigma, \mathbf{b})$**  as follows.

- For every given  $x \in [0, \infty)$ , to find a probability measure  $\mathbb{Q}^\bullet$  on the space  $(\Omega_1, \mathcal{F}_1)$ , under which:
- (i)  $\omega_1(0) = x$  and  $\int_0^\infty \mathbf{1}_{\{\omega_1(t)=0\}} dt = 0$  hold  $\mathbb{Q}^\bullet$ -a.e.; and moreover,
  - (ii) for every function  $\psi \in C^2([0, \infty); \mathbb{R})$  with  $\psi'(0+) \geq 0$ , the process  $K^\psi(\cdot)$  is a continuous local submartingale, and a continuous local martingale whenever  $\psi'(0+) = 0$ , with respect to the filtration  $\mathbb{F}_1^\bullet = \{\mathcal{F}_1^\bullet(t)\}_{0 \leq t < \infty}$  with  $\mathcal{F}_1^\bullet(t) := \mathcal{F}_1^\circ(t+)$ .

Here we have denoted by  $\mathbb{F}_1^\circ := \{\mathcal{F}_1^\circ(t), 0 \leq t < \infty\}$  the augmentation of  $\mathbb{F}_1$  under  $\mathbb{Q}^\bullet$ . As usual, we shall say that this problem is *well-posed*, if it admits exactly one solution. For a recent study of the well-posedness of submartingale problems for obliquely reflected diffusions, in domains with piecewise smooth boundaries, see KANG & RAMANAN (2014), where a random measure similar to (2.19) is derived.

## 6.2 A Local Martingale Problem for the Planar Diffusion

Next, we consider the canonical space  $\Omega_2 := C([0, \infty); \mathbb{R}^2)$  of  $\mathbb{R}^2$ -valued continuous functions on  $[0, \infty)$  endowed with the  $\sigma$ -algebra  $\mathcal{F}_2 := \mathcal{B}(\Omega_2)$  of its BOREL sets. We consider also its coördinate mapping and the natural filtration  $\mathbb{F}_2 := \{\mathcal{F}_2(t)\}_{0 \leq t < \infty}$  with  $\mathcal{F}_2(t) = \sigma(\omega_2(s), 0 \leq s \leq t)$ . We recall, here and in what follows, the Definitions 4.1 and 4.2.

Given a probability measure  $\mu$  on the BOREL subsets of the unit circumference  $\mathfrak{S}$ , and BOREL-measurable functions  $\mathbf{b} : [0, \infty) \rightarrow \mathbb{R}$ ,  $\sigma : [0, \infty) \rightarrow \mathbb{R} \setminus \{0\}$  as in subsection 6.1, we define for every function  $g \in \mathfrak{D}$  the process

$$M^g(\cdot; \omega_2) := g(\omega_2(\cdot)) - g(\omega_2(0)) - \int_0^\cdot \mathcal{L}g(\omega_2(t)) \cdot \mathbf{1}_{\{\|\omega_2(t)\| > 0\}} dt, \quad \text{where} \quad (6.2)$$

$$\mathcal{L}g(x) := \mathbf{b}(\|x\|) G'(x) + \frac{1}{2} \mathbf{a}(\|x\|) G''(x); \quad x \in \mathbb{R}^2.$$

### 6.2.1 The Local Martingale Problem

Motivated by the generalized FREIDLIN-SHEU formula (4.5) in Theorem 4.1, we formulate now the **Local Martingale Problem associated with the triple  $(\sigma, \mathbf{b}, \mu)$**  as follows.

- For every fixed  $x \in \mathbb{R}^2$ , to find a probability measure  $\mathbb{Q}$  on the canonical space  $(\Omega_2, \mathcal{F}_2)$ , such that:
- (i)  $\omega_2(0) = x$  holds  $\mathbb{Q}$ -a.e.;
  - (ii) the analogue of the “non-stickiness” property (2.17) holds, namely

$$\int_0^\infty \mathbf{1}_{\{\omega_2(t)=0\}} dt = 0, \quad \mathbb{Q} - \text{a.e.}; \quad (6.3)$$

- (iii) for every function  $g$  in  $\mathfrak{D}_+^\mu$  (respectively,  $\mathfrak{D}^\mu$ ) as in (4.4), the process  $M^g(\cdot; \omega_2)$  of (6.2) is a continuous local submartingale (resp., martingale) with respect to the filtration  $\mathbb{F}_2^\bullet := \{\mathcal{F}_2^\bullet(t)\}_{0 \leq t < \infty}$ .

Here we have set  $\mathcal{F}_2^\bullet(t) := \mathcal{F}_2^\circ(t+)$ , and denoted by  $\mathbb{F}_2^\circ = \{\mathcal{F}_2^\circ(t)\}_{0 \leq t < \infty}$  the  $\mathbb{Q}$ -augmentation of the filtration  $\mathbb{F}_2$ . Again, this problem is called “well-posed” if it admits exactly one solution.

• The theory of the STROOCK & VARADHAN martingale problem is extended in Proposition 6.1 right below, for a continuous planar semimartingale  $X(\cdot)$  that satisfies the properties (2.17), (2.18) and, with coefficients  $\gamma_i$ ,  $i = 1, 2$  given through (2.11) and (2.10), the system of stochastic integral equations

$$X_i(\cdot) = X_i(0) + \int_0^\cdot f_i(X(t)) \left[ \mathbf{b}(\|X(t)\|) dt + \boldsymbol{\sigma}(\|X(t)\|) dW(t) \right] + \gamma_i L^{\|X\|}(\cdot), \quad i = 1, 2. \quad (6.4)$$

**Proposition 6.1. Stochastic Equations for WALSH Diffusions:** (a) For every weak solution  $(X, W)$ ,  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$  to the system of stochastic equations (6.4), we have

$$\|X(\cdot)\| = \|X(0)\| + \int_0^\cdot \mathbf{1}_{\{\|X(t)\| > 0\}} \left( \mathbf{b}(\|X(t)\|) dt + \boldsymbol{\sigma}(\|X(t)\|) dW(t) \right) + L^{\|X\|}(\cdot); \quad (6.5)$$

and if this weak solution also satisfies the conditions (2.17)-(2.18), then it induces a solution to the local martingale problem associated with the triple  $(\boldsymbol{\sigma}, \mathbf{b}, \boldsymbol{\mu})$ .

(b) Conversely, every solution to the local martingale problem associated with the triple  $(\boldsymbol{\sigma}, \mathbf{b}, \boldsymbol{\mu})$  induces a weak solution to the system (6.4) which satisfies the properties (2.17), (2.18). The state process  $X(\cdot)$  in this weak solution solves also the system of stochastic equations (2.12) with “folded driver”  $S(\cdot) = \|X(\cdot)\|$ .

(c) Uniqueness holds for the local martingale problem associated with  $(\boldsymbol{\sigma}, \mathbf{b}, \boldsymbol{\mu})$ , if and only if uniqueness in distribution holds for the system of (6.4) subject to the conditions (2.17), (2.18).

*Proof of Part (a):* We first validate (6.5) for any weak solution to (6.4). From (6.4) we see

$$\int_0^T (|f_i(X(t)) \mathbf{b}(\|X(t)\|)| + f_i^2(X(t)) \boldsymbol{\alpha}(\|X(t)\|)) dt < \infty, \quad i = 1, 2, \quad 0 \leq T < \infty.$$

Since  $f_1^2(x) + f_2^2(x) = 1$  and  $|f_1(x)| + |f_2(x)| \geq 1$  hold for any  $x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ , we obtain then

$$\int_0^T \mathbf{1}_{\{\|X(t)\| > 0\}} (|\mathbf{b}(\|X(t)\|)| + \boldsymbol{\alpha}(\|X(t)\|)) dt < \infty, \quad 0 \leq T < \infty. \quad (6.6)$$

Let us recall the stopping time  $\sigma_\varepsilon = \inf\{t > 0 : \|X(t)\| \leq \varepsilon\}$  for every  $\varepsilon > 0$ . Since the function  $x \mapsto \|x\| = \sqrt{x_1^2 + x_2^2}$  is smooth on  $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ , we get the following from (6.4) by ITÔ’s formula:

$$\|X(\cdot \wedge \sigma_\varepsilon)\| = \|X(0)\| + \int_0^{\cdot \wedge \sigma_\varepsilon} \left( \mathbf{b}(\|X(t)\|) dt + \boldsymbol{\sigma}(\|X(t)\|) dW(t) \right). \quad (6.7)$$

With  $\tau(0) := \inf\{t \geq 0 : \|X(t)\| = 0\} = \lim_{\varepsilon \downarrow 0} \sigma_\varepsilon$ , we let  $\varepsilon \downarrow 0$  in (6.7) and obtain (from (6.6))

$$\|X(\cdot \wedge \tau(0))\| = \|X(0)\| + \int_0^\cdot \mathbf{1}_{(0, \tau(0))}(t) \left( \mathbf{b}(\|X(t)\|) dt + \boldsymbol{\sigma}(\|X(t)\|) dW(t) \right).$$

Recall now the stopping times  $\{\tau_m^\varepsilon, m \in \mathbb{N}_0\}$  defined in (5.6). In the same manner as above we obtain

$$\|X(\cdot \wedge \tau_{2\ell+2}^\varepsilon)\| - \|X(\cdot \wedge \tau_{2\ell+1}^\varepsilon)\| = \int_0^\cdot \mathbf{1}_{(\tau_{2\ell+1}^\varepsilon, \tau_{2\ell+2}^\varepsilon)}(t) \left( \mathbf{b}(\|X(t)\|) dt + \boldsymbol{\sigma}(\|X(t)\|) dW(t) \right)$$

for  $\ell \in \mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$  with  $\tau_{-1}^\varepsilon \equiv 0$ . We decompose  $\|X(T)\|$  as in the proof of Theorem 4.1:

$$\|X(T)\| = \|X(0)\| + \sum_{\ell \in \mathbb{N}_{-1}} (\|X(T \wedge \tau_{2\ell+2}^\varepsilon)\| - \|X(T \wedge \tau_{2\ell+1}^\varepsilon)\|) + \sum_{\ell \in \mathbb{N}_0} (\|X(T \wedge \tau_{2\ell+1}^\varepsilon)\| - \|X(T \wedge \tau_{2\ell}^\varepsilon)\|). \quad (6.8)$$

With the previous considerations, letting  $\varepsilon \downarrow 0$ , we obtain (6.5) for the radial process  $\|X(\cdot)\|$ . The continuous semimartingale  $X(\cdot)$  thus solves also the system (2.12) with the “folded driver”  $S(\cdot) = \|X(\cdot)\|$ .

Suppose now that the properties (2.17)-(2.18) are also satisfied by the weak solution we have posited. Thanks to Theorem 4.1, for every given function  $g \in \mathfrak{D}_+^\mu$  (resp.,  $g \in \mathfrak{D}^\mu$ ), the process  $M^g(\cdot; X)$  as in (6.2) is then a local submartingale (resp., martingale). The property (6.3) comes from (2.17). Consequently, a solution  $\mathbb{Q}$  to the local martingale problem associated with the triple  $(\sigma, \mathbf{b}, \mu)$  is given by the probability measure  $\mathbb{Q} = \mathbb{P}X^{-1}$  induced by the process  $X(\cdot)$  on the canonical space  $(\Omega_2, \mathcal{F}_2)$ .  $\square$

*Proof of Part (b):* Conversely, suppose that the local martingale problem associated with the triple  $(\sigma, \mathbf{b}, \mu)$  has a solution  $\mathbb{Q}$ . We recall the notation in (2.11) and define on the canonical space the processes

$$X(\cdot) \equiv (X_1(\cdot), X_2(\cdot))' := (\|\omega_2(\cdot)\| \mathbf{f}_1(\omega_2(\cdot)), \|\omega_2(\cdot)\| \mathbf{f}_2(\omega_2(\cdot)))', \quad (6.9)$$

$$M_i(\cdot) := X_i(\cdot) - X_i(0) - \int_0^\cdot \mathbf{b}(\|X(t)\|) \mathbf{f}_i(X(t)) dt - \gamma_i \left( \|X(\cdot)\| - \|X(0)\| - \int_0^\cdot \mathbf{b}(\|X(t)\|) \mathbf{1}_{\{\|X(t)\| > 0\}} dt \right), \quad (6.10)$$

$$M_{i,k}(\cdot) := g_{i,k}(X(\cdot)) - g_{i,k}(X(0)) - 2 \int_0^\cdot \|X(t)\| \mathbf{b}(\|X(t)\|) (\mathbf{f}_i(X(t)) - \gamma_i)(\mathbf{f}_k(X(t)) - \gamma_k) \mathbf{1}_{\{\|X(t)\| > 0\}} dt \\ - \int_0^\cdot \mathbf{a}(\|X(t)\|) (\mathbf{f}_i(X(t)) - \gamma_i)(\mathbf{f}_k(X(t)) - \gamma_k) \mathbf{1}_{\{\|X(t)\| > 0\}} dt$$

for  $1 \leq i, k \leq 2$ , as well as

$$M_{i,i}^\circ(\cdot) := X_i^2(\cdot) - X_i^2(0) - \int_0^\cdot \mathbf{f}_i^2(X(t)) \left[ 2 \|X(t)\| \mathbf{b}(\|X(t)\|) + \mathbf{a}(\|X(t)\|) \right] dt. \quad (6.11)$$

• Here, as in Proposition 4.1, we consider the following functions in the family  $\mathfrak{D}^\mu$  of (4.4):

$$g_1(x) := r(\cos(\theta) - \gamma_1), \quad g_2(x) := r(\sin(\theta) - \gamma_2), \quad g_{i,k}(x) := g_i(x) g_k(x) \quad (6.12)$$

for  $x = (r, \theta) \in \mathbb{R}^2$ ,  $1 \leq i, k \leq 2$ . We consider also the functions  $g_{i,i}^\circ \in \mathfrak{D}^\mu$  and  $g_3 \in \mathfrak{D}_+^\mu$  defined by

$$g_{1,1}^\circ(x) := r^2 \cos^2(\theta), \quad g_{2,2}^\circ(x) := r^2 \sin^2(\theta), \quad g_3(x) := r; \quad x \in \mathbb{R}^2. \quad (6.13)$$

• We deduce then from (6.2) that the processes  $M_i(\cdot) \equiv M^{g_i}(\cdot; X)$ ,  $M_{i,k}(\cdot) \equiv M^{g_{i,k}}(\cdot; X)$  as well as  $M_{i,i}^\circ(\cdot) \equiv M^{g_{i,i}^\circ}(\cdot; X)$ , are continuous local martingales for  $1 \leq i, k \leq 2$ ; and that so are the processes

$$M_{i,k}(\cdot) - g_i(X(0))M_k(\cdot) - g_k(X(0))M_i(\cdot) - \int_0^\cdot \left( \int_0^t (\mathbf{f}_k(X(s)) - \gamma_k) \mathbf{b}(\|X(s)\|) \mathbf{1}_{\{\|X(s)\| > 0\}} ds \right) dM_i(t) \\ - \int_0^\cdot \left( \int_0^t (\mathbf{f}_i(X(s)) - \gamma_i) \mathbf{b}(\|X(s)\|) \mathbf{1}_{\{\|X(s)\| > 0\}} ds \right) dM_k(t) = M_i(\cdot)M_k(\cdot) - \int_0^\cdot r_{i,k}(t) dt.$$

This way, we identify for  $1 \leq i, k \leq 2$  the cross-variation structure

$$\langle M_i, M_k \rangle(\cdot) = \int_0^\cdot r_{i,k}(t) dt, \quad r_{i,k}(t) := \mathbf{a}(\|X(t)\|) (\mathbf{f}_i(X(t)) - \gamma_i)(\mathbf{f}_k(X(t)) - \gamma_k) \mathbf{1}_{\{\|X(t)\| > 0\}}. \quad (6.14)$$

• We also observe that the continuous process

$$N(\cdot) := M^{g_3}(\cdot; X) = \|X(\cdot)\| - \|X(0)\| - \int_0^\cdot \mathbf{b}(\|X(t)\|) \mathbf{1}_{\{\|X(t)\| > 0\}} dt \quad (6.15)$$

is a local submartingale; this way we obtain the semimartingale property of the radial process  $\|X(\cdot)\|$ . By the DOOB-MEYER decomposition (e.g., KARATZAS & SHREVE (1991), Theorem 1.4.10), there exists then an adapted, continuous and increasing process  $A(\cdot)$  such that

$$M_3(\cdot) := N(\cdot) - A(\cdot) = \|X(\cdot)\| - \|X(0)\| - \int_0^\cdot \mathbf{b}(\|X(t)\|) \mathbf{1}_{\{\|X(t)\| > 0\}} dt - A(\cdot) \quad (6.16)$$

is a continuous local martingale. We claim that this increasing process is  $A(\cdot) = L^{\|X\|}(\cdot)$ , the local time at the origin of the continuous, nonnegative semimartingale  $\|X(\cdot)\|$ .



• In order to substantiate this claim, let us fix two arbitrary positive constants  $c_1, c_2$  with  $c_1 < c_2$  and define a sequence of stopping times inductively, via  $\sigma_0 := \inf\{t \geq 0 : \|X(t)\| = c_2\}$  if  $\|X(0)\| < c_2$  and  $\sigma_0 := 0$  otherwise; as well as

$$\sigma_{2n+1} := \inf\{t \geq \sigma_{2n} : \|X(t)\| = c_1\}, \quad \sigma_{2n+2} := \inf\{t \geq \sigma_{2n+1} : \|X(t)\| = c_2\}; \quad n \in \mathbb{N}_0.$$

We note that  $\|X(t)\| \geq c_1$  holds for  $t \in (\sigma_{2n}, \sigma_{2n+1})$ ; and conversely, that  $\|X(t)\| > c_2$  implies  $t \in (\sigma_{2n}, \sigma_{2n+1})$  for some  $n \in \mathbb{N}_0$ . Thus, by taking an appropriate smooth function  $g_4 \in \mathfrak{D}^\mu$  of the form  $g_4(r, \theta) = \psi(r)$  where  $\psi : [0, \infty) \rightarrow [0, \infty)$  is smooth with  $\psi(r) = r$  for  $r \geq c_1$ , one can show that  $N(\cdot \wedge \sigma_{2n+1}) - N(\cdot \wedge \sigma_{2n})$  is a continuous local martingale.

Then, since both processes  $N(\cdot \wedge \sigma_{2n+1}) - A(\cdot \wedge \sigma_{2n+1})$  and  $N(\cdot \wedge \sigma_{2n}) - A(\cdot \wedge \sigma_{2n})$  are continuous local martingales, so is  $A(\cdot \wedge \sigma_{2n+1}) - A(\cdot \wedge \sigma_{2n})$ . But this last process is of bounded variation, so  $A(\cdot \wedge \sigma_{2n+1}) \equiv A(\cdot \wedge \sigma_{2n})$  for every  $n \in \mathbb{N}_0$ . In other words, the process  $A(\cdot)$  is flat on  $[\sigma_{2n}, \sigma_{2n+1}]$  for every  $n$ . Therefore we have  $\int_0^\infty \mathbf{1}_{\{\|X(t)\| \in (c_2, \infty)\}} dA(t) \equiv 0$ , because  $\|X(t)\| \in (c_2, \infty)$  implies  $t \in (\sigma_{2n}, \sigma_{2n+1})$  for some  $n \in \mathbb{N}_0$ . Since  $c_2 > 0$  can be chosen arbitrarily small, we obtain

$$A(\cdot) = \int_0^\cdot \mathbf{1}_{\{\|X(t)\|=0\}} dA(t), \quad \text{and} \quad \int_0^\cdot \|X(t)\| dA(t) = 0. \quad (6.17)$$

In conjunction with (6.15)–(6.17), the characterization  $L^{\|X\|}(\cdot) = \int_0^\cdot \mathbf{1}_{\{\|X(t)\|=0\}} d\|X(t)\|$  for the local time of a continuous, nonnegative semimartingale such as  $\|X(\cdot)\|$ , establishes then the claim, since

$$L^{\|X\|}(\cdot) = \int_0^\cdot \mathbf{1}_{\{\|X(t)\|=0\}} \left( \mathbf{b}(\|X(t)\|) \mathbf{1}_{\{\|X(t)\|>0\}} dt + dA(t) \right) = A(\cdot).$$

• We return to the computation of the cross-variations  $\langle M_i, M_k \rangle(\cdot)$  for  $1 \leq i, k \leq 3$ . Recalling (6.16), (6.17) and  $\langle \|X\| \rangle(\cdot) = \langle M_3 \rangle(\cdot) = \langle N \rangle(\cdot)$ , an application of Itô's rule to  $\|X(\cdot)\|^2$  gives

$$\|X(\cdot)\|^2 - \|X(0)\|^2 - 2 \int_0^\cdot \|X(t)\| \mathbf{b}(\|X(t)\|) \mathbf{1}_{\{\|X(t)\|>0\}} dt - \langle N \rangle(\cdot) = 2 \int_0^\cdot \|X(t)\| dM_3(t).$$

Combining the last identity with (6.11), we observe that

$$M_{1,1}^\circ(\cdot) + M_{2,2}^\circ(\cdot) - 2 \int_0^\cdot \|X(t)\| dM_3(t) = \langle N \rangle(\cdot) - \int_0^\cdot \mathbf{a}(\|X(t)\|) \mathbf{1}_{\{\|X(t)\|>0\}} dt \quad (6.18)$$

is both a local martingale and a continuous process of bounded variation; thus we identify

$$\langle \|X\| \rangle(\cdot) = \langle N \rangle(\cdot) = \langle M_3 \rangle(\cdot) = \int_0^\cdot r_{3,3}(t) dt \quad \text{where} \quad r_{3,3}(t) := \mathbf{a}(\|X(t)\|) \mathbf{1}_{\{\|X(t)\|>0\}}. \quad (6.19)$$

By analogy with the derivation of (6.18), and taking (6.10) into account, we observe that

$$M_{i,i}^\circ(\cdot) - 2 \int_0^\cdot X_i(t) dM_i(t) - 2\gamma_i \int_0^\cdot X_i(t) dM_3(t) = \langle X_i \rangle(\cdot) - \int_0^\cdot \mathbf{a}(\|X(t)\|) [\mathbf{f}_i(X(t))]^2 dt$$

is both a local martingale and a continuous process of bounded variation for  $i = 1, 2$ ; thus we identify

$$\langle X_i \rangle(\cdot) = \int_0^\cdot \mathbf{a}(\|X(t)\|) [\mathbf{f}_i(X(t))]^2 dt, \quad i = 1, 2. \quad (6.20)$$

It follows from (6.9) that  $\langle M_i \rangle(\cdot) = \langle X_i \rangle(\cdot) + \gamma_i^2 \langle \|X\| \rangle(\cdot) - 2\gamma_i \langle X_i, \|X\| \rangle(\cdot)$ ; and in conjunction with (6.19), (6.20), (6.14), this gives  $\langle X_i, \|X\| \rangle(\cdot) = \int_0^\cdot \mathbf{1}_{\{X(t) \neq 0\}} \mathbf{a}(\|X(t)\|) \mathbf{f}_i(X(t)) dt$ . Hence, with  $r_{i,3}(t) \equiv r_{3,i}(t) := \mathbf{a}(\|X(t)\|) (\mathbf{f}_i(X(t)) - \gamma_i) \mathbf{1}_{\{X(t) \neq 0\}}$  for  $i = 1, 2$ , we obtain

$$\langle M_i, M_3 \rangle(\cdot) \equiv \langle M_3, M_i \rangle(\cdot) = \langle X_i, \|X\| \rangle(\cdot) - \gamma_i \langle \|X\| \rangle(\cdot) = \int_0^\cdot r_{i,3}(t) dt.$$

- We have now computed all elements of the  $(3 \times 3)$  matrix  $(d\langle M_i, M_k \rangle(t) / dt)_{1 \leq i, k \leq 3} = (r_{i,k}(t))_{1 \leq i, k \leq 3}$ ; we observe also that this matrix is of rank 1, on  $\{t \geq 0 : X(t) \neq \mathbf{0}\}$ . By Theorem 3.4.2 and Proposition 5.4.6 in KARATZAS & SHREVE (1991), there exists an extension of the original probability space, and on it
  - (i) a three-dimensional standard Brownian motion  $\widetilde{W}(\cdot) = (\widetilde{W}_1(\cdot), \widetilde{W}_2(\cdot), \widetilde{W}_3(\cdot))'$ ,
  - (ii) a one-dimensional standard Brownian motion  $W(\cdot)$ , and
  - (iii) measurable, adapted, matrix-valued processes  $(\rho_{i,k}(\cdot))_{1 \leq i, k \leq 3}$  with  $\int_0^T [\rho_{i,k}(t)]^2 dt < \infty$ ,
 such that we have the representations

$$M_i(\cdot) = \sum_{k=1}^3 \int_0^\cdot \rho_{i,k}(t) d\widetilde{W}_k(t) = \int_0^\cdot \sigma(\|X(t)\|) (f_i(X(t)) - \gamma_i) \mathbf{1}_{\{X(t) \neq \mathbf{0}\}} dW(t), \quad i = 1, 2 \quad (6.21)$$

and  $M_3(\cdot) = \int_0^\cdot \sigma(\|X(t)\|) \mathbf{1}_{\{X(t) \neq \mathbf{0}\}} dW(t)$ . Substituting this into the decomposition  $N(\cdot) = M_3(\cdot) + L^{\|X\|}(\cdot)$  and then into (6.15), we obtain the stochastic equation (6.5) for the radial process  $\|X(\cdot)\|$ . Substituting the expressions of (6.21), (6.5) into  $M_i(\cdot)$  in (6.9) for  $i = 1, 2$ , we observe that the process  $X(\cdot)$  defined in (6.9) satisfies the system of (6.4). It follows from (6.3) that  $X(\cdot)$  satisfies the property (2.17).

- Finally, for every set  $A \in \mathcal{B}([0, 2\pi))$ , we consider the functions

$$g_5(x) := g_5(r, \theta) = r(\mathbf{1}_A(\theta) - \nu(A)) \quad \text{and} \quad g_6(x) := g_6(r, \theta) = r \mathbf{1}_A(\theta) \quad (6.22)$$

in polar coördinates. Since  $g_5 \in \mathfrak{D}^\mu$  and  $g_6 \in \mathfrak{D}_+^\mu$  we obtain that the process

$$g_5(X(\cdot)) - g_5(X(0)) - \int_0^\cdot b(\|X(t)\|) (\mathbf{1}_{\{\arg(X(t)) \in A\}} - \nu(A)) \mathbf{1}_{\{\|X(t)\| > 0\}} dt \quad (6.23)$$

is a continuous local martingale, and that the process

$$g_6(X(\cdot)) - g_6(X(0)) - \int_0^\cdot b(\|X(t)\|) \mathbf{1}_{\{\arg(X(t)) \in A\} \cap \{\|X(t)\| > 0\}} dt$$

is a continuous local submartingale.

Repeating an argument similar to the one deployed above, we identify  $\nu(A)L^{\|X\|}(\cdot)$  as the local time  $L^{R^A}(\cdot)$  at the origin for the continuous, non-negative semimartingale  $R^A(\cdot) := g_6(X(\cdot))$ . Indeed,

$$R^A(\cdot) - R^A(0) - \int_0^\cdot b(\|X(t)\|) \mathbf{1}_{\{\arg(X(t)) \in A\} \cap \{\|X(t)\| > 0\}} dt - L^{R^A}(\cdot) \quad (6.24)$$

is a continuous local martingale. Moreover, on account of (6.16), we see that

$$\nu(A) \left( \|X(\cdot)\| - \|X(0)\| - \int_0^\cdot b(\|X(t)\|) \mathbf{1}_{\{\|X(t)\| > 0\}} dt - L^{\|X\|}(\cdot) \right) \quad (6.25)$$

is also a continuous local martingale. Subtracting (6.24) from (6.23) and adding (6.25), we deduce that the finite variation process  $L^{R^A}(\cdot) - \nu(A)L^{\|X\|}(\cdot)$  is a continuous local martingale, and hence identically zero, i.e.,  $L^{R^A}(\cdot) \equiv \nu(A)L^{\|X\|}(\cdot)$  as in (2.18).

We conclude from this analysis, that the system of equations (6.4) admits a weak solution with the properties (2.17) and (2.18). This proves Part (b). Part (c) is now evident.  $\square$

*Remark 6.1.* Looking back to the definition of the above local martingale problem for the planar diffusion, we recall Definition 4.1 and observe that the following statements (i)-(ii) are equivalent:

- (i) For every  $g \in \mathfrak{D}_+^\mu$ , the process  $M^g(\cdot; \omega_2)$  is a continuous local submartingale;
- (ii) For every  $g \in \mathfrak{D}^\mu$ , the process  $M^g(\cdot; \omega_2)$  is a continuous local martingale, and the process  $M^{g_3}(\cdot; \omega_2)$  is a continuous local submartingale, where  $g_3(x) = \|x\| = r$  is defined in (6.13).

If (i) holds,  $M^{g_3}(\cdot; \omega_2)$  is a continuous local submartingale since  $g_3(x) = \|x\|$  belongs to  $\mathfrak{D}_+^\mu$ . For every  $g \in \mathfrak{D}^\mu$  we have  $g \in \mathfrak{D}_+^\mu$  and  $-g \in \mathfrak{D}_+^\mu$ , hence both  $M^g(\cdot; \omega_2)$  and  $M^{-g}(\cdot; \omega_2) = -M^g(\cdot; \omega_2)$  are continuous local submartingales. Thus  $M^g(\cdot; \omega_2)$  is a continuous local martingale, and (ii) follows.

Next, let us assume (ii). Every  $g \in \mathfrak{D}_+^\mu$  can then be decomposed as  $g := g_{(1)} + g_{(2)}$ , where  $g_{(1)}(x) = c\|x\|$  with  $c := \int_0^{2\pi} g'_\theta(0+) \nu(d\theta) \geq 0$  and  $g_{(2)}(\cdot) := g(\cdot) - g_{(1)}(\cdot) \in \mathfrak{D}^\mu$ . Thus the above condition (ii) implies that  $M^{g_{(1)}}(\cdot; \omega_2) = c\|\omega_2(\cdot)\|$  is a local submartingale and  $M^{g_{(2)}}(\cdot; \omega_2)$  is a local martingale, and hence  $M^g(\cdot; \omega_2) = M^{g_{(1)}}(\cdot; \omega_2) + M^{g_{(2)}}(\cdot; \omega_2)$  is a local submartingale, and (i) follows.

*Remark 6.2.* In the proof of Proposition 6.1(b) let us define a random measure  $\mathbf{R}(t, dz)$  on  $\mathcal{B}(\mathfrak{S})$  for every  $t \geq 0$  with  $\mathbf{R}(t, dz) \equiv \mathbf{R}(t, d\theta)$  for  $\theta \in [0, 2\pi)$ ,  $z \in \mathfrak{S}$  via  $g_6(X(t)) = R^A(t) = \int_A \mathbf{R}(t, d\theta)$ ,  $A \in \mathcal{B}([0, 2\pi))$ . Then for every  $t \geq 0$  we may write  $X(t) = \int_{\mathfrak{S}} z \mathbf{R}(t, dz)$ .

### 6.3 Well-Posedness

We conjecture that, if the local submartingale problem associated with the pair  $(\sigma, \mathbf{b})$  is well-posed, then the same is true for the local martingale problem associated with the triple  $(\sigma, \mathbf{b}, \mu)$ .

The result that follows settles this conjecture in the affirmative, for the driftless case  $\mathbf{b} \equiv \mathbf{0}$ . Proposition 6.3 then deals with the case of a drift  $\mathbf{b} = \sigma \mathbf{c}$  with  $\mathbf{c} : \mathbb{R}_+ \rightarrow \mathbb{R}$  bounded and measurable.

**Proposition 6.2. Well-Posedness for Driftless WALSH Diffusions:** *Suppose that*

(i) *the drift  $\mathbf{b}$  is identically equal to zero; and that*

(ii) *the reciprocal of the dispersion coefficient  $\sigma : [0, \infty) \rightarrow \mathbb{R} \setminus \{0\}$  is locally square-integrable; i.e.,*

$$\int_K \frac{dy}{\sigma^2(y)} < \infty \quad \text{holds for every compact set } K \subset [0, \infty). \quad (6.26)$$

*Then the local submartingale problem of subsection 6.1, associated with the pair  $(\sigma, \mathbf{0})$ , is well-posed.*

*Moreover, the local martingale problem of subsection 6.2 associated with the triple  $(\sigma, \mathbf{0}, \mu)$  is also well-posed; and uniqueness in distribution holds, subject to the properties in (2.17) and (2.18), for the corresponding system of stochastic integral equations in (6.4) with  $\mathbf{b} \equiv \mathbf{0}$ , namely,*

$$X_i(\cdot) = X_i(0) + \int_0^\cdot \mathbf{f}_i(X(t)) \sigma(\|X(t)\|) dW(t) + \gamma_i L^{\|X\|}(\cdot), \quad i = 1, 2. \quad (6.27)$$

*Proof of Existence:* Let us consider the stochastic integral equation

$$S(\cdot) = r + \int_0^\cdot \sigma(S(t)) dW(t) + L^S(\cdot) \quad (6.28)$$

driven by one-dimensional Brownian motion  $W(\cdot)$ . It is shown in SCHMIDT (1989) that, under (6.26), this equation (6.28) has a non-negative, unique-in-distribution weak solution; equivalently, the STROOCK & VARADHAN (1971) local submartingale problem associated with  $(\sigma, \mathbf{0})$  for  $K^\psi(\cdot)$  in (6.1) is well-posed.

Let us also verify the property (2.4). From Exercise 3.7.10 in KARATZAS & SHREVE (1991), we get

$$0 = \int_0^\infty \mathbf{1}_{\{S(t)=0\}} d\langle S \rangle(t) = \int_0^\infty \mathbf{1}_{\{S(t)=0\}} \sigma^2(S(t)) dt, \quad \text{thus also} \quad \int_0^\infty \mathbf{1}_{\{S(t)=0\}} dt = 0$$

because  $\sigma(\cdot)$  never vanishes. It follows then from Theorem 2.1 that, on a suitably enlarged probability space, we may construct from this reflected diffusion  $S(\cdot)$  a continuous, planar semimartingale  $X(\cdot)$  which satisfies  $\|X(\cdot)\| = S(\cdot)$ , the system of equations (6.27), and the properties (2.14)–(2.18). On the strength of Proposition 6.1(a), the local martingale problem associated with the triple  $(\sigma, \mathbf{0}, \mu)$  admits a solution.  $\square$

*Proof of Uniqueness:* We adopt the idea of proof in Theorem 3.2 of BARLOW, PITMAN & YOR (1989). Suppose there are two solutions  $\mathbb{Q}_j$ ,  $j = 1, 2$  to this local martingale problem associated with the triple  $(\sigma, \mathbf{0}, \mu)$ . Let us take an arbitrary set  $A \in \mathcal{B}([0, 2\pi))$  and consider the functions  $h_A(\cdot)$  and  $g_A(\cdot)$  defined as in (4.6) for the indicator  $\varphi = \mathbf{1}_A$ , namely

$$h_A(x) := (\mathbf{1}_{\{\arg(x) \in A\}} - \nu(A)) \cdot \mathbf{1}_{\{\|x\| > 0\}} = (\nu(A^c) \mathbf{1}_{\{\arg(x) \in A\}} - \nu(A) \mathbf{1}_{\{\arg(x) \in A^c\}}) \cdot \mathbf{1}_{\{\|x\| > 0\}}, \quad (6.29)$$

$$g_A(x) := \|x\| h_A(x), \quad x \in \mathbb{R}^2. \quad (6.30)$$

The above function  $g_A(\cdot)$  belongs to the family  $\mathfrak{D}^\mu$  in (4.4), as does the function  $[g_A(\cdot)]^2$ . By assumption and Proposition 4.1, the process  $M_A(\cdot) := g_A(\omega_2(\cdot))$  is then a  $\mathbb{Q}_j$ -local martingale, with

$$\langle M_A \rangle(T) = \langle g_A(\omega_2(\cdot)) \rangle(T) = \int_0^T [h_A(\omega_2(t))]^2 \mathbf{a}(\|\omega_2(t)\|) dt; \quad 0 \leq T < \infty, \quad j = 1, 2.$$

Note that  $\omega_2(\cdot)$  and  $\|\omega_2(\cdot)\|$  solve in the weak sense the equations (6.27) and (6.28), respectively. The argument at the beginning of Section 3 implies that  $\omega_2(\cdot)$  stays on the same ray on each of its excursion away from the origin. Moreover,  $\|\omega_2(\cdot)\|$  is strongly Markovian with respect to the filtration  $\mathbb{F}_2$ , and its distribution is uniquely determined.

• Let us assume  $0 < \nu(A) < 1$  first. We note that  $g_A(x) > 0$ , if  $\arg(x) \in A$ ;  $g_A(x) < 0$  if  $\arg(x) \in A^c$ ; and  $g_A(x) = 0$  if  $x = \mathbf{0}$ . It is also easy to verify that the process

$$U_A(\cdot) := \int_0^\cdot \left( \frac{1}{\nu(A^c)} \cdot \mathbf{1}_{\{g_A(\omega_2(t)) > 0\}} + \frac{1}{\nu(A)} \cdot \mathbf{1}_{\{g_A(\omega_2(t)) \leq 0\}} \right) \cdot \frac{dM_A(t)}{\sigma(\|\omega_2(t)\|)} \quad (6.31)$$

is a continuous  $\mathbb{Q}_j$ -local martingale with  $\langle U_A \rangle(t) = t$  for  $t \geq 0$ ; i.e., a  $\mathbb{Q}_j$ -Brownian motion for  $j = 1, 2$ . The probability distribution of the process  $M_A(\cdot) = g_A(\omega_2(\cdot))$  is then determined uniquely and *independently of the solution  $\mathbb{Q}_j$ ,  $j = 1, 2$  to the local martingale problem*. This is because, under the assumption (6.26) on the dispersion coefficient and thanks to the theory of ENGELBERT & SCHMIDT (1984), the stochastic differential equation driven by the Brownian motion  $U_A(\cdot)$  and derived from (6.31),

$$dM_A(t) = \varrho(M_A(t)) dU_A(t), \quad 0 \leq t < \infty \quad (6.32)$$

with  $c_0 := \nu(A^c)$ ,  $c_1 := \nu(A)$  and the new dispersion function

$$\varrho(x) := c_0 \cdot \sigma\left(\frac{x}{c_0}\right) \cdot \mathbf{1}_{\{x > 0\}} + c_1 \cdot \sigma\left(-\frac{x}{c_1}\right) \cdot \mathbf{1}_{\{x \leq 0\}}; \quad x \in \mathbb{R}, \quad (6.33)$$

admits a weak solution, which is unique in the sense of the probability distribution. This follows from Theorem 5.5.7 in KARATZAS & SHREVE (1991), and from the fact that the reciprocal of the function  $\varrho(\cdot)$  inherits the local square-integrability property (6.26) of the reciprocal of  $\sigma(\cdot)$ . Moreover,  $M_A(\cdot) = g_A(\omega_2(\cdot))$  is strongly Markovian with respect to the filtration  $\mathbb{F}_2$  (cf. the proof of Lemma 9.2 in Section 9). Therefore, for an arbitrary  $C \in \mathcal{B}((0, \infty))$ , recalling (6.29) and (6.30), we have

$$\begin{aligned} \mathbb{Q}_j(\|\omega_2(t)\| \in C, \arg(\omega_2(t)) \in A \mid \mathcal{F}_2(s)) &= \mathbb{Q}_j(g_A(\omega_2(t)) \in \nu(A^c)C \mid \mathcal{F}_2(s)) \\ &= \mathbb{Q}_j(g_A(\omega_2(t)) \in \nu(A^c)C \mid g_A(\omega_2(s))), \quad 0 \leq s < t < \infty, \quad j = 1, 2. \end{aligned}$$

Since the distribution of the process  $g_A(\omega_2(\cdot))$  is uniquely determined, the above probability does not depend on  $j = 1, 2$ . We conclude then that  $\mathbb{Q}_j(\|\omega_2(t)\| \in C, \arg(\omega_2(t)) \in A \mid \mathcal{F}_2(s))$  does not depend on  $j = 1, 2$ , if  $0 < \nu(A) < 1$ .

• For the resulting diffusion process in natural scale, we shall denote by

$$\begin{aligned} p_A(s, t; y, B) &:= \mathbb{Q}_j(g_A(\omega_2(t)) \in B \mid g_A(\omega_2(s)) = y) \\ p_A^*(s, t; y, B) &:= \mathbb{Q}_j(g_A(\omega_2(t)) \in B, \tau_s(\omega_2) > t \mid g_A(\omega_2(s)) = y) \\ q_A(s, t; y) &:= p_A^*(s, t; y, \mathbb{R}) := \mathbb{Q}_j(\tau_s(\omega_2) > t \mid g_A(\omega_2(s)) = y) \end{aligned}$$

its transition, taboo-transition, and survival probabilities (for both  $j = 1, 2$  on the strength of uniqueness in distribution for (6.32)). Here  $0 \leq s < t < \infty$ ,  $y \in \mathbb{R}$  and  $B \in \mathcal{B}$  are arbitrary, and we have denoted the first hitting time of the origin by

$$\tau_s(\omega_2) := \inf \{u \geq s : \|\omega_2(u)\| = 0\} = \inf \{u \geq s : g_A(\omega_2(u)) = 0\}$$

For an arbitrary  $C \in \mathcal{B}((0, \infty))$  and recalling (6.29), (6.30), we have then the expression

$$\mathbb{Q}_j(\|\omega_2(t)\| \in C, \arg(\omega_2(t)) \in A, \tau_s(\omega_2) > t \mid \omega_2(s) = x) = p_A^*(s, t; \nu(A^c)\|x\|, \nu(A^c)C) \mathbf{1}_A(\arg(x)),$$

whose right-hand side does not depend on  $j = 1, 2$ . Similarly, we observe that the transition probability

$$\mathbb{Q}_j(\|\omega_2(t)\| \in C, \arg(\omega_2(t)) \in A, \tau_s(\omega_2) < t \mid \omega_2(s) = x)$$

is given, with  $m := \nu(A^c) \mathbf{1}_A(\arg(x)) - \nu(A) \mathbf{1}_{A^c}(\arg(x))$ , by the expression

$$\begin{aligned} & \int_s^t \mathbb{Q}_j(\|\omega_2(t)\| \in C, \arg(\omega_2(t)) \in A \mid \tau_s(\omega_2) = \theta, \omega_2(s) = x) \cdot \mathbb{Q}_j(\tau_s(\omega_2) \in d\theta \mid \omega_2(s) = x) \\ &= \int_s^t p_A(\theta, t; 0, \nu(A^c)C) \cdot \left( -d_\theta q_A(s, \theta; m \|x\|) \right). \end{aligned}$$

Once again, the expression on the right-hand side does not depend on  $j = 1, 2$ .

• Next, we consider the case  $\nu(A) \in \{0, 1\}$ . Let  $\nu(A) = 0$  first; then  $g_A(x) = \|x\| \mathbf{1}_{\{\arg(x) \in A, \|x\| > 0\}}$ , and the process  $M_A(\cdot) = g_A(\omega_2(\cdot))$  is a nonnegative, continuous  $\mathbb{Q}_j$ -local martingale, thus also a supermartingale – so it stays at the origin  $\mathbf{0}$  after hitting it for the first time. It follows that with  $\mathbb{Q}_j$ -probability one, the angular part  $\arg(\omega_2(\cdot))$  never again visits the set  $A$ , after the radial part  $\|\omega_2(\cdot)\|$  first becomes zero. Thus for an arbitrary  $C \in \mathcal{B}((0, \infty))$  and for every  $0 \leq s < t < \infty$ ,  $x \in \mathbb{R}^2$ ,  $j = 1, 2$  we have

$$\mathbb{Q}_j(\|\omega_2(t)\| \in C, \arg(\omega_2(t)) \in A, \tau_s(\omega_2) < t \mid \mathcal{F}_2(s)) = 0.$$

If, on the other hand,  $\nu(A) = 1$  holds, then  $\nu(A^c) = 0$  and therefore

$$\mathbb{Q}_j(\|\omega_2(t)\| \in C, \arg(\omega_2(t)) \in A^c, \tau_s(\omega_2) < t \mid \mathcal{F}_2(s)) = 0,$$

which implies

$$\begin{aligned} \mathbb{Q}_j(\|\omega_2(t)\| \in C, \arg(\omega_2(t)) \in A, \tau_s(\omega_2) < t \mid \mathcal{F}_2(s)) &= \mathbb{Q}_j(\|\omega_2(t)\| \in C, \tau_s(\omega_2) < t \mid \mathcal{F}_2(s)) \\ &= \mathbb{Q}_j(\|\omega_2(t)\| \in C, \tau_s(\omega_2) < t \mid \|\omega_2(s)\|). \end{aligned}$$

We have also the following in both cases:

$$\begin{aligned} \mathbb{Q}_j(\|\omega_2(t)\| \in C, \arg(\omega_2(t)) \in A, \tau_s(\omega_2) > t \mid \mathcal{F}_2(s)) \\ = \mathbb{Q}_j(\|\omega_2(t)\| \in C, \tau_s(\omega_2) > t \mid \|\omega_2(s)\|) \mathbf{1}_A(\arg(\omega_2(s))). \end{aligned}$$

Since the distribution of  $\|\omega_2(\cdot)\|$  is uniquely determined and independent of  $j = 1, 2$ , we conclude that  $\mathbb{Q}_j(\|\omega_2(t)\| \in C, \arg(\omega_2(t)) \in A \mid \mathcal{F}_2(s))$  does not depend on  $j = 1, 2$ , if  $\nu(A) = 0$  or  $1$ .

• Finally, we note that  $\mathbb{Q}_j(\omega_2(t) = 0 \mid \omega_2(s) = x) = \mathbb{Q}_j(\|\omega_2(t)\| = 0 \mid \|\omega_2(s)\| = \|x\|)$ , where the right-hand side is also uniquely determined.

• It is clear that the conditional distribution of  $\omega_2(t)$  given  $\mathcal{F}_2(s)$  is uniquely determined for  $0 \leq s < t < \infty$ . Standard arguments show then, that the finite-dimensional distributions of  $\omega_2(\cdot)$  are uniquely determined. Therefore, the local martingale problem of the Proposition is well-posed.  $\square$

**Proposition 6.3. Well-Posedness for WALSH Diffusions with Drift:** *Under the setting of Proposition 6.2, and in addition to the assumptions imposed there, let us consider another function  $c : \mathbb{R}_+ \rightarrow \mathbb{R}$  which is bounded and measurable. We denote by  $\mathbb{Q}^{(0)}$  the solution to the local martingale problem of subsection 6.2 associated with the triple  $(\sigma, \mathbf{0}, \mu)$ .*

(i) *For every  $T \in (0, \infty)$ , the local martingale problem associated with the triple  $(\sigma, \sigma c, \mu)$  for  $M^g(t)$ ,  $0 \leq t \leq T$  in (6.2), is then well posed, and its solution is given by the probability measure  $\mathbb{Q}_T^{(c)}$  with*

$$\left. \frac{d\mathbb{Q}_T^{(c)}}{d\mathbb{Q}^{(0)}} \right|_{\mathcal{F}_2^\bullet(t)} := \exp \left( \int_0^t c(\|\omega_2(u)\|) dW(u) - \frac{1}{2} \int_0^t c^2(\|\omega_2(u)\|) du \right); \quad 0 \leq t \leq T.$$

(ii) Under the assumptions in (i), suppose that  $\mathbb{Q}^{(c)}$  solves the local martingale problem associated with the triple  $(\sigma, \sigma c, \mu)$ . Then there exists an  $\mathbb{F}_2$ -Brownian motion  $B(\cdot)$ , such that every  $\mathbb{F}_2$ -local martingale  $M(\cdot)$  with  $M(0) = 0$  can be represented in the integral form  $M(\cdot) = \int_0^\cdot H(t) dB(t)$  for some  $\mathbb{F}_2$ -progressively measurable and locally square-integrable process  $H(\cdot)$ .

*Proof.* (i) This is a direct consequence of Propositions 6.1, 6.2 and GIRSANOV's change of measure. Indeed, it follows from Proposition 6.1 that, under  $\mathbb{Q}^{(0)}$ , the coordinate process  $\omega_2(\cdot)$  satisfies the system of stochastic integral equations (6.27), subject to (2.17) and (2.18). Because of the boundedness of the function  $c(\cdot)$ , the measure  $\mathbb{Q}_T^{(c)}$  just introduced is a probability.

By GIRSANOV's theorem (e.g., KARATZAS & SHREVE (1991), Theorem 3.5.1) we see that for every fixed  $T \in (0, \infty)$ , the process  $W^{(c)}(u) := W(u) - \int_0^u c(\|\omega_2(t)\|) dt$ ,  $0 \leq u \leq T$  is standard Brownian motion under this probability measure  $\mathbb{Q}_T^{(c)}$ , and thus the coordinate process  $\omega_2(\cdot)$  satisfies on the time-horizon  $[0, T]$  the system of stochastic integral equations

$$X_i(\cdot) = x + \int_0^\cdot f_i(X(t)) \sigma(\|X(t)\|) \left[ dW^{(c)}(t) + c(\|X(t)\|) dt \right] + \gamma_i L^{\|X\|}(\cdot), \quad i = 1, 2.$$

Moreover, since the probability measure  $\mathbb{Q}_T^{(c)}$  is absolutely continuous with respect to  $\mathbb{Q}^{(0)}$ , we obtain (2.17) and (2.18) with  $X(\cdot)$  replaced by  $\omega_2(\cdot)$ , a.e. under  $\mathbb{Q}_T^{(c)}$ . Thanks to Proposition 6.1 again,  $\mathbb{Q}_T^{(c)}$  solves the local martingale problem of subsection 6.2 associated with the triple  $(\sigma, \sigma c, \mu)$ .

Conversely, for any solution  $\mathbb{Q}_T^{(c)}$  to the local martingale problem associated with  $(\sigma, \sigma c, \mu)$  for  $M^g(t)$ ,  $0 \leq t \leq T$  as in (6.2), the probability measure  $\mathbb{Q}^{(0)}$  defined via

$$\frac{d\mathbb{Q}^{(0)}}{d\mathbb{Q}_T^{(c)}} \Big|_{\mathcal{F}_2^\bullet(t)} := \exp \left( - \int_0^t c(\|\omega_2(u)\|) dW^{(c)}(u) - \frac{1}{2} \int_0^t c^2(\|\omega_2(u)\|) du \right); \quad 0 \leq t \leq T$$

is seen to solve the local martingale problem of subsection 6.2 associated with the triple  $(\sigma, 0, \mu)$ . Since this problem is well-posed, the same holds for the local martingale problem associated with  $(\sigma, \sigma c, \mu)$ .

(ii) From part (i) we know that  $\mathbb{Q}^{(c)}|_{\mathcal{F}_2^\bullet(T)} = \mathbb{Q}_T^{(c)}$ ,  $0 \leq T < \infty$ , and  $B(\cdot) := W^{(c)}(\cdot)$  is a standard Brownian motion under  $\mathbb{Q}^{(c)}$ . Since the local martingale problem in part (i) is well-posed, we can adapt the proof of Theorem 4.1 of BARLOW, PITMAN & YOR (1989) to show that the required  $H(\cdot)$  exists up to any finite time  $T$  (see also JACOD (1977)), and can thus be defined on all of  $[0, \infty)$ .  $\square$

## 7 Martingale Characterization of the WALSH Brownian Motion

We still have to show that, when  $U(\cdot) \equiv B(\cdot)$  is standard Brownian motion, the construction of Theorem 2.1 leads to the WALSH Brownian motion as defined, for instance, in BARLOW, PITMAN & YOR (1989) or FITZSIMMONS & KUTER (2014). In the present section we establish this connection; cf. Proposition 7.2.

Following these sources, we may characterize the WALSH Brownian motion  $\mathbf{W}(\cdot)$  in terms of its FELLER semigroup  $\{\mathcal{P}_t, t \geq 0\}$  defined for  $f \in C_0(\overline{E})$  via

$$\begin{aligned} [\mathcal{P}_t f](0, \theta) &:= T_t^+ \overline{f}(0), \\ [\mathcal{P}_t f](r, \theta) &:= T_t^+ \overline{f}(r) + [T_t^0(f_\theta - \overline{f})](r); \quad r > 0, \theta \in [0, 2\pi). \end{aligned} \tag{7.1}$$

Here  $\{T_t^+, 0 \leq t < \infty\}$  is the semigroup of reflected Brownian motion on  $[0, \infty)$ , and  $\{T_t^0, 0 \leq t < \infty\}$  the semigroup of Brownian motion on  $[0, \infty)$  killed upon reaching the origin. For the sake of simplicity, we use polar coordinates in the punctured plane  $E$  of (2.8). Abusing notation slightly, we define also

$$\overline{f}(r) := \int_{[0, 2\pi)} f(r, \theta) \nu(d\theta), \quad f_\theta(r) := f(r, \theta); \quad (r, \theta) \in \overline{E}, \tag{7.2}$$

for  $f \in C(\overline{E})$ , as in (2.9). Let us assume that  $\mathbf{W}(0) = \mathbf{x} \in \mathbb{R}^2$ . BARLOW, PITMAN & YOR (1989) show that there is a FELLER and strong MARKOV process  $\mathbf{W}(\cdot)$  with values in  $\mathbb{R}^2$ , continuous paths, and  $\{\mathcal{P}_t, 0 \leq t < \infty\}$  as its semigroup. This is the process these authors call “WALSH Brownian motion”. They show that the radial part  $\|\mathbf{W}(\cdot)\|$  is one-dimensional reflecting Brownian motion. For this planar process  $\mathbf{W}(\cdot)$ , HAJRI & TOUHAMI (2014) derive a version of the FREIDLIN-SHEU formula, that involves the standard, one-dimensional Brownian motion of the filtration  $\mathbb{F}^{\mathbf{W}} = \{\mathcal{F}^{\mathbf{W}}(t)\}_{0 \leq t < \infty}$ , given by

$$\beta^{\mathbf{W}}(\cdot) := \|\mathbf{W}(\cdot)\| - \|\mathbf{x}\| - L^{\|\mathbf{W}\|}(\cdot). \quad (7.3)$$

Here is an extension of Proposition 3.1 in BARLOW, PITMAN & YOR (1989); we recall the Definitions 4.1 and 4.2, as well as the notation of (4.6). It shows that the WALSH Brownian motion with spinning measure  $\mu$ , defined via the semigroup (7.1), generates a solution to the local martingale problem associated with the triple  $(1, 0, \mu)$  (cf. Remark 6.1).

**Proposition 7.1. Properties of WALSH Brownian Motion:** *Let  $\mathbf{W}(\cdot)$  be the WALSH Brownian motion defined via the semigroup (7.1) and with spinning measure  $\mu$ . Then:*

- (i) *The process  $\|\mathbf{W}(\cdot)\|$  is reflecting Brownian motion; and  $\mathbf{W}(\cdot)$  satisfies the properties in (2.14)–(2.15).*
- (ii) *For any  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  in the class  $\mathfrak{D}^\mu$  of (4.4), the continuous process below is a local martingale:*

$$g(\mathbf{W}(\cdot)) - g(\mathbf{x}) - \frac{1}{2} \int_0^\cdot G''(\mathbf{W}(t)) \mathbf{1}_{\{\mathbf{W}(t) \neq 0\}} dt = \int_0^\cdot G'(\mathbf{W}(t)) \mathbf{1}_{\{\mathbf{W}(t) \neq 0\}} d\beta^{\mathbf{W}}(t).$$

*Proof:* The claims of (i) are proved in BARLOW, PITMAN & YOR (1989). Claim (ii) follows by applying the FREIDLIN-SHEU-type formula of Theorem 1.2 in HAJRI & TOUHAMI (2014) to the process  $g(\mathbf{W}(\cdot))$ . We also note that, with the notation of (4.6), both processes below are continuous martingales:

$$\mathcal{M}_{(\varphi)}^{\mathbf{W}}(\cdot) = g_{(\varphi)}(\mathbf{W}(\cdot)) - g_{(\varphi)}(\mathbf{x}) = \int_0^\cdot h_{(\varphi)}(\mathbf{W}(t)) d\beta^{\mathbf{W}}(t), \quad \mathcal{N}_{(\varphi)}^{\mathbf{W}}(\cdot) = (\mathcal{M}_{(\varphi)}^{\mathbf{W}}(\cdot))^2 - \langle \mathcal{M}_{(\varphi)}^{\mathbf{W}} \rangle(\cdot). \quad \square$$

Our next result shows that, as we expected all along, WALSH semimartingales driven by Brownian motions  $U(\cdot)$  are WALSH Brownian motions defined via the semigroup (7.1).

**Proposition 7.2. Stochastic Equations for WALSH Brownian Motions:** *Let us place ourselves in the context of Theorem 2.1, and suppose that the semimartingale  $U(\cdot) \equiv B(\cdot)$  of (2.1) is Brownian motion. Then the planar process  $X(\cdot)$  constructed there, has the following properties:*

- (i) *It is the unique-in-distribution weak solution, subject to the properties (2.17), (2.18), of the system of stochastic integral equations in (3.4), namely  $X_i(\cdot) = x_i + \int_0^\cdot f_i(X(t)) dB(t) + \gamma_i L^{\|X\|}(\cdot)$ ,  $i = 1, 2$ .*
- (ii) *It is a WALSH Brownian motion.*
- (iii) *Every  $\mathbb{F}^X$ -local martingale  $M(\cdot)$  with  $M(0) = 0$  has an integral representation  $M(\cdot) = \int_0^\cdot H(t) dB(t)$ , for some  $\mathbb{F}^X$ -progressively measurable and locally square-integrable process  $H(\cdot)$ .*

*Proof.* The first claim follows from Propositions 6.1, 6.2 with  $\sigma(\cdot) \equiv 1$ ; the second claim, that  $X(\cdot)$  is WALSH Brownian motion, is a consequence of Propositions 6.1, 6.2 and 7.1. With  $U(\cdot) \equiv B(\cdot)$  a standard Brownian motion, Proposition 4.1 shows that both processes below are continuous local martingales

$$M_{(\varphi)}(\cdot) = g_{(\varphi)}(X(\cdot)) - g_{(\varphi)}(\mathbf{x}) = \int_0^\cdot h_{(\varphi)}(X(t)) dB(t), \quad N_{(\varphi)}(\cdot) = [M_{(\varphi)}(\cdot)]^2 - \int_0^\cdot [h_{(\varphi)}(X(t))]^2 dt$$

(cf. Theorem 3.1 of BARLOW, PITMAN & YOR (1989)). The third claim follows from Proposition 6.3.  $\square$

In the terminology adopted by MANSUY & YOR (2006), and for a spinning measure  $\mu$  that does not concentrate on one or two points in  $\mathfrak{S}$ , this last property says that the natural filtration  $\mathbb{F}^X$  of the WALSH Brownian motion is a *weak Brownian filtration* (has the martingale representation property with respect to  $U$ ) but *not a strong Brownian filtration* (cannot be generated by a Brownian motion of any dimension).

## 8 Angular Dependence

Let us admit now bounded, BOREL-measurable coefficients  $\mathbf{b} : \mathbb{R} \times [0, 2\pi) \rightarrow \mathbb{R}$  and  $\mathbf{a} : \mathbb{R} \times [0, 2\pi) \rightarrow (0, \infty)$  which may depend on the angular variable  $\theta \in [0, 2\pi)$  in (6.2). We assume also that  $\mathbf{a}$  is bounded away from zero, and consider the local martingale problem of subsection 6.2 but now with the infinitesimal generator re-defined as

$$\mathcal{L}^* g(x) := \mathbf{b}(\|x\|, \arg(x)) G'(x) + \frac{1}{2} \mathbf{a}(\|x\|, \arg(x)) G''(x); \quad x \in \mathbb{R}^2, \quad g \in \mathfrak{D}. \quad (8.1)$$

For every given, fixed  $\theta \in [0, 2\pi)$ , we set  $\sigma_\theta(r) := \sigma(r, \theta)$  as well as  $\mathbf{a}(r, \theta) = [\sigma(r, \theta)]^2$ , and define the scale function  $p_\theta(\cdot)$  by

$$p_\theta(r) = p(r, \theta) := \int_0^r \exp\left(-2 \int_0^\xi \frac{\mathbf{b}(\zeta, \theta)}{\mathbf{a}(\zeta, \theta)} d\zeta\right) d\xi, \quad r \in [0, \infty),$$

as well as its inverse  $q_\theta(r) = q(r, \theta)$  in the radial component with  $q_\theta(p_\theta(r)) = r$ . We note that these functions satisfy  $p_\theta(0) = 0 = q_\theta(0)$  and  $p'_\theta(0+) = 1 = q'_\theta(0+)$ ; that  $p_\theta(\cdot)$  has an absolutely continuous, strictly positive derivative; that the second derivative  $p''_\theta(\cdot)$  exists almost everywhere; and that both of these derivatives are bounded. Therefore, by the generalized ITô rule, we see that Theorem 4.1 holds also for the function  $p_\theta(\cdot)$ , which may not be in the class  $\mathfrak{D}$ ; the same is true for the function  $q_\theta(\cdot)$ .

Let us consider an auxiliary diffusion coefficient

$$\tilde{\sigma}_\theta(r) \equiv \tilde{\sigma}(r, \theta) := p'_\theta(q_\theta(r)) \sigma_\theta(q_\theta(r)), \quad 0 < r < \infty \quad (8.2)$$

and  $\theta \in [0, 2\pi)$ , and write  $\tilde{\sigma}(y) \equiv \tilde{\sigma}(r, \theta)$  for  $y = (r, \theta) \in \mathbb{R}^2$ . We introduce also the stochastic clock

$$\mathcal{Q}(\cdot) := \int_0^\cdot \frac{du}{[\tilde{\sigma}(\|X(u)\|, \Theta(u))]^2} \quad \text{and its inverse} \quad \mathcal{T}(t) := \inf\{v \geq 0 : \mathcal{Q}(v) > t\}; \quad 0 \leq t < \infty.$$

Here  $X(\cdot) = Z(\cdot)S(\cdot)$  is a WALSH semimartingale as constructed as in (5.2), starting from a one-dimensional Brownian motion  $U(\cdot) = B(\cdot)$  in Proposition 7.2. In particular,  $X(\cdot)$  is WALSH Brownian motion; whereas  $\Theta(\cdot) = \arg(X(\cdot))$  is as in (5.5). We consider now the time-changed, rescaled version  $Y(\cdot) = (Y_1(\cdot), Y_2(\cdot))'$  of this WALSH Brownian motion  $X(\cdot)$ , defined in polar coördinates via

$$\|Y(\cdot)\| := q(\|X(\mathcal{T}(\cdot))\|, \arg(X(\mathcal{T}(\cdot)))) , \quad \arg(Y(\cdot)) := \arg(X(\mathcal{T}(\cdot))) = \Theta(\mathcal{T}(\cdot)). \quad (8.3)$$

In terms of this rescaling, we have the representation

$$\mathcal{T}(\cdot) = \int_0^\cdot \left( p'_\theta(r) \sigma_\theta(r) \right)^2 \Big|_{\theta=\arg(Y(t)), r=\|Y(t)\|} dt \quad (8.4)$$

for the inverse clock. The resulting process  $Y(\cdot)$  turns out to be a WALSH semimartingale with angular dependence in its local characteristics  $(\sigma, \mathbf{b}, \mu)$ , as follows.

**Proposition 8.1.** *The process  $Y(\cdot)$  defined in (8.3) satisfies the integral equations*

$$Y(\cdot) = Y(0) + \int_0^\cdot \mathfrak{f}(Y(t)) [\sigma(\|Y(t)\|, \arg(Y(t))) dW(t) + \mathbf{b}(\|Y(t)\|, \arg(Y(t))) dt] + \gamma L^{\|Y\|}(\cdot) \quad (8.5)$$

$$\|Y(\cdot)\| = \|Y(0)\| + \int_0^\cdot \mathbf{1}_{\{\|Y(t)\| > 0\}} (\mathbf{b}(\|Y(t)\|, \arg(Y(t))) dt + \sigma(\|Y(t)\|, \arg(Y(t))) dW(t)) + L^{\|Y\|}(\cdot)$$

as well as the properties  $\int_0^\cdot \mathbf{1}_{\{Y(t)=0\}} dt \equiv 0$  and  $L^{R_*^A}(\cdot) \equiv \nu(A) L^{\|Y\|}(\cdot)$ ,  $\forall A \in \mathcal{B}([0, 2\pi))$ . Furthermore, it induces a solution to the local martingale problem associated with the triple  $(\sigma, \mathbf{b}, \mu)$  and  $\mathcal{L}^*$  in (8.1).

In the above expressions  $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2)'$  is defined in (2.8),  $W(\cdot)$  is one-dimensional Brownian motion, and the “thinned” process  $R_*^A(\cdot) := \|Y(\cdot)\| \cdot \mathbf{1}_A(\arg(Y(\cdot)))$  is defined for  $A \in \mathcal{B}([0, 2\pi))$ .



*Proof:* Applying the FREIDLIN-SHEU formula in Theorem 4.1 to  $\mathbf{q}(X(\cdot))$ , we obtain

$$\begin{aligned} \|Y(\cdot)\| &= \mathbf{q}(X(\mathcal{T}(\cdot))) = \mathbf{q}(x) + \int_0^{\mathcal{T}(\cdot)} \mathbf{Q}'(X(u)) \mathbf{1}_{\{X(u) \neq \mathbf{0}\}} dB(u) + \left( \int_0^{2\pi} \mathbf{q}'_{\theta}(0+) \nu(d\theta) \right) \cdot L^{\|X\|}(\mathcal{T}(\cdot)) \\ &\quad + \frac{1}{2} \int_0^{\mathcal{T}(\cdot)} \mathbf{Q}''(X(u)) \mathbf{1}_{\{X(u) \neq \mathbf{0}\}} du. \end{aligned} \quad (8.6)$$

Here by direct calculation

$$\mathbf{Q}'(x) := \mathbf{q}'_{\theta}(r) = \frac{1}{\mathbf{p}'_{\theta}(\mathbf{q}_{\theta}(r))}, \quad \mathbf{Q}''(x) := \mathbf{q}''_{\theta}(r) = \frac{2\mathbf{b}(\mathbf{q}_{\theta}(r), \theta)}{\mathbf{a}(\mathbf{q}_{\theta}(r), \theta) \cdot (\mathbf{p}'_{\theta}(\mathbf{q}_{\theta}(r)))^2} \quad (8.7)$$

hold for every  $x = (r, \theta)$ , where  $r$  is not in a set of Lebesgue measure zero that depends on  $\theta \in [0, 2\pi)$ . Thanks to the P. LÉVY Theorem, the continuous local martingale

$$W(\cdot) := \int_0^{\mathcal{T}(\cdot)} \frac{dB(u)}{\tilde{\sigma}(\|X(u)\|, \arg(X(u)))}$$

is one-dimensional standard Brownian motion. Since  $\text{Leb}(\{t : X(t) = \mathbf{0}\}) = \text{Leb}(\{t : S(t) = 0\}) = 0$  a.s. and  $\mathbf{q}(\mathbf{0}) = 0$  from the construction, we obtain

$$\text{Leb}(\{t : \|Y(t)\| = \mathbf{q}(X(\mathcal{T}(t))) = \mathbf{0}\}) = \text{Leb}(\mathcal{T}^{-1}\{t : X(t) = \mathbf{0}\}) = 0 \text{ a.s.} \quad (8.8)$$

In conjunction with the definitions (8.2) and (8.3), we obtain now the representations

$$\begin{aligned} \int_0^{\mathcal{T}(\cdot)} \mathbf{Q}'(X(u)) \mathbf{1}_{\{X(u) \neq \mathbf{0}\}} dB(u) &= \int_0^{\cdot} \mathbf{Q}'(X(\mathcal{T}(u))) \mathbf{1}_{\{X(\mathcal{T}(u)) \neq \mathbf{0}\}} dB(\mathcal{T}(u)) \\ &= \int_0^{\cdot} \mathbf{1}_{\{\|Y(u)\| > 0\}} \boldsymbol{\sigma}(\|Y(u)\|, \arg(Y(u))) dW(u) \end{aligned} \quad (8.9)$$

(on the strength of Proposition 3.4.8 in KARATZAS & SHREVE (1991)), as well as

$$\begin{aligned} \int_0^{\mathcal{T}(\cdot)} \mathbf{Q}''(X(u)) \mathbf{1}_{\{X(u) \neq \mathbf{0}\}} du &= \int_0^{\cdot} \mathbf{Q}''(X(\mathcal{T}(u))) \frac{d\mathcal{T}(u)}{du} \mathbf{1}_{\{X(\mathcal{T}(u)) \neq \mathbf{0}\}} du \\ &= 2 \int_0^{\cdot} \mathbf{1}_{\{\|Y(u)\| > 0\}} \mathbf{b}(\|Y(u)\|, \arg(Y(u))) du \end{aligned} \quad (8.10)$$

(by time-change). From these considerations and (8.8) we also obtain the identification of local time

$$L^{\|Y\|}(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{Y(u)=0\}} d\|Y\|(u) = \left( \int_0^{2\pi} \mathbf{q}'_{\theta}(0+) \nu(d\theta) \right) \cdot L^{\|X\|}(\mathcal{T}(\cdot)), \quad (8.11)$$

thus also the dynamics for the radial part of the process  $Y(\cdot)$ , namely

$$\|Y(\cdot)\| = \|Y(0)\| + \int_0^{\cdot} \mathbf{1}_{\{\|Y(t)\| > 0\}} (\mathbf{b}(\|Y(t)\|, \arg(Y(t))) dt + \boldsymbol{\sigma}(\|Y(t)\|, \arg(Y(t))) dW(t)) + L^{\|Y\|}(\cdot).$$

• Recalling (8.3), and applying the FREIDLIN-SHEU formula in Theorem 4.1 to the process  $Y_i(\cdot) = \mathbf{q}(X(\mathcal{T}(\cdot))) \mathbf{f}_i(X(\mathcal{T}(\cdot)))$ , we obtain

$$\begin{aligned} Y_i(\cdot) &= y_i + \int_0^{\mathcal{T}(\cdot)} \mathbf{Q}'(X(u)) \mathbf{f}_i(X(u)) \mathbf{1}_{\{X(u) \neq \mathbf{0}\}} dB(u) + \frac{1}{2} \int_0^{\mathcal{T}(\cdot)} \mathbf{Q}''(X(u)) \mathbf{f}_i(X(u)) \mathbf{1}_{\{X(u) \neq \mathbf{0}\}} du \\ &\quad + \left( \int_0^{2\pi} \mathbf{q}'_{\theta}(0+) \cos\left(\theta - \frac{\pi}{2}(i-1)\right) \nu(d\theta) \right) L^{\|X\|}(\mathcal{T}(\cdot)); \quad i = 1, 2. \end{aligned}$$

Hence, combining with (8.9)-(8.11) and  $\mathbf{f}(\mathbf{0}) = 0$  and  $\mathbf{q}'_{\theta}(0+) = 1$ , we obtain the dynamics (8.5).

• Furthermore, for every  $g \in \mathfrak{D}$  by another application of the FREIDLIN-SHEU formula in Theorem 4.1 to  $g(Y(\cdot)) = g(\mathbf{q}(r, \theta), \theta) \big|_{r=\|X(\mathcal{T}(\cdot))\|, \theta=\arg(X(\mathcal{T}(\cdot)))}$  with  $\mathbf{q}_\theta(0+) = 0$ , we derive

$$\begin{aligned} g(Y(T)) &= g(y) + \int_0^T \mathbf{1}_{\{Y(t) \neq \mathbf{0}\}} (\mathbf{b}(\|Y(t)\|, \arg(Y(t))) G'(Y(t)) + \frac{1}{2} \mathbf{a}(\|Y(t)\|, \arg(Y(t))) G''(Y(t))) dt \\ &+ \int_0^T \mathbf{1}_{\{Y(t) \neq \mathbf{0}\}} G'(Y(t)) \boldsymbol{\sigma}(\|Y(t)\|, \arg(Y(t))) dW(t) + \left( \int_0^{2\pi} g'_\theta(0+) \boldsymbol{\nu}(d\theta) \right) \cdot L^{\|Y\|}(T), \quad 0 \leq T < \infty, \end{aligned} \quad (8.12)$$

in conjunction with (8.7)-(8.11) and  $\mathbf{q}'_\theta(0+) = 1$ . When  $g \in \mathfrak{D}^\mu$ , we can apply this to  $M^g(\cdot; Y)$  in (6.2) – now redefined with the operator  $\mathcal{L}^*$  of (8.1) – to conclude that  $M^g(\cdot; Y)$  is equal to the local martingale

$$g(Y(\cdot)) - g(y) - \int_0^\cdot \mathcal{L}^* g(Y(t)) \mathbf{1}_{\{Y(t) \neq \mathbf{0}\}} dt = \int_0^\cdot G'(Y(t)) \mathbf{1}_{\{Y(t) \neq \mathbf{0}\}} \boldsymbol{\sigma}(\|Y(t)\|, \arg(Y(t))) dW(t).$$

Therefore, the local martingale problem associated with the triple  $(\boldsymbol{\sigma}, \mathbf{b}, \boldsymbol{\mu})$  and the second-order differential operator  $\mathcal{L}^*$  in (8.1), is seen to have a solution. The properties of  $Y(\cdot)$  are now verified readily.  $\square$

**Proposition 8.2.** *With the assumptions and notation of this section, the local martingale problem of subsection 6.2, associated with the triple  $(\boldsymbol{\sigma}, \mathbf{b}, \boldsymbol{\mu})$  and the operator  $\mathcal{L}^*$  in (8.1), is well-posed.*

*Proof:* Existence of a solution to this local martingale problem is established by Proposition 8.1.

To prove uniqueness, we can reverse the steps of the construction in Proposition 8.1, as follows. Consider any solution of the local martingale problem of subsection 6.2, associated with the triple and  $(\boldsymbol{\sigma}, \mathbf{b}, \boldsymbol{\mu})$  and the operator  $\mathcal{L}^*$ , and the coördinate process  $Y(\cdot) := \omega_2(\cdot)$  on the canonical space for that problem. We introduce the time change  $\mathcal{T}(\cdot)$  as in (8.4), along with its inverse  $\mathcal{Q}(\cdot)$ ; as well as the time-changed, rescaled version  $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$  of the process  $Y(\cdot)$ , defined in polar coördinates via

$$\|X(\cdot)\| := \mathbf{p}(\|Y(\mathcal{Q}(\cdot))\|, \arg(Y(\mathcal{Q}(\cdot)))) , \quad \arg(X(\cdot)) := \arg(Y(\mathcal{Q}(\cdot))) . \quad (8.13)$$

Using Proposition 6.1 (rather, its obvious generalization to coefficients with angular dependence) and Theorem 4.1, we have for the planar process  $Y(\cdot)$  the appropriate FREIDLIN-SHEU-formula. With this at hand, the planar process  $X(\cdot)$  is seen to be a WALSH Brownian motion with spinning measure  $\boldsymbol{\mu}$ , in a manner similar to that in the proof of Proposition 8.1. The path  $t \mapsto X(t)$  is, with probability one, continuous in the topology induced by the tree metric (3.3), and hence so is the path  $t \mapsto Y(t)$ . In terms of this WALSH Brownian motion, we can express the time change  $\mathcal{Q}(\cdot)$  as

$$\mathcal{Q}(\cdot) = \int_0^\cdot \frac{du}{[\tilde{\boldsymbol{\sigma}}(\|X(u)\|, \arg(X(u)))]^2} .$$

The crucial step now, is to note that the process  $Y(\cdot)$  can be written as  $Y(t) = \Psi_t(X(\cdot))$ . Here  $\Psi_\cdot$  is a measurable mapping defined by  $\Psi_t(\omega_2) = \mathbf{q}(\Pi_{\mathcal{T}(t; \omega_2)}(\omega_2))$ , in terms of the measurable projection mapping  $\Pi_t(\omega_2) := \omega_2(t)$  and the continuous time change

$$\mathcal{T}(t; \omega_2) := \inf \left\{ v \geq 0 : \int_0^v \frac{du}{[\tilde{\boldsymbol{\sigma}}(\|\omega_2(u)\|, \arg(\omega_2(u)))]^2} > t \right\}, \quad 0 \leq t < \infty .$$

Since the distribution of the WALSH Brownian motion  $X(\cdot)$  is uniquely determined (see section 7), the distribution of  $Y(\cdot)$  is also determined uniquely from these considerations.

We conclude that the local martingale problem associated with the triple  $(\boldsymbol{\sigma}, \mathbf{b}, \boldsymbol{\mu})$  is well-posed.  $\square$

## 9 The Time-Homogeneous Strong MARKOV Property

From section 7, we know that the unique solution to the well-posed local martingale problem associated with the triple  $(1, 0, \mu)$  induces a WALSH Brownian motion, which is a time-homogeneous strong MARKOV process as shown in BARLOW, PITMAN & YOR (1989). We generalize this result in subsection 9.1, by showing that every solution to a well-posed local martingale problem as in subsection 6.2, associated with a triple  $(\sigma, b, \mu)$ , induces a time-homogeneous strong MARKOV process.

Next, we pick up the thread of Proposition 6.1(a), and try to see what we can say about solutions to *the system of stochastic equations (6.4) for given  $(\gamma_1, \gamma_2) \in \mathbb{R}^2$ , subject only to the non-stickiness condition (2.17)*. We find that for some such solutions there is no spinning measure  $\mu$  such that the “thinning condition” (2.18) is satisfied. We show that the time-homogeneous strong MARKOV property can be used to rule out these solutions. Then for every solution with an appropriate version of this property, we prove the existence of a spinning measure  $\mu$  for which the local martingale problem associated with the triple  $(\sigma, b, \mu)$  is solved by the distribution of the state process  $X(\cdot)$  in the solution. In this spirit we obtain in subsection 9.2 a similar conclusion as in Part (a) of Proposition 6.1, but with the notable difference that here  $\mu$  is not given in advance; its existence is established in the proof of Theorem 9.1, the next major result of this work. As a corollary of this result, we show in subsection 9.3 that with  $b = 0$ ,  $\sigma = 1$  the equations (6.4), subject to (2.17) and to the time-homogeneous strong MARKOV property, characterize WALSH Brownian motions.

Throughout this section, we shall always refer to subsection 6.1 for local submartingale problems associated with pairs  $(\sigma, b)$  (corresponding to one-dimensional reflected diffusions), and to subsection 6.2 for local martingale problems associated with triples  $(\sigma, b, \mu)$  (corresponding to planar diffusions).

### 9.1 On Well-posed Local Martingale Problems

**Definition 9.1.** Given a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$ , we say that a progressively measurable process  $X(\cdot)$  with values in some Euclidean space  $\mathbb{R}^d$  is *time-homogeneous strongly Markovian* with respect to it if, for every stopping time  $T$  of  $\mathbb{F}$ , real number  $t \geq 0$ , and set  $\Gamma \in \mathcal{B}(\mathbb{R}^d)$ ,

$$\mathbb{P}(X(T+t) \in \Gamma \mid \mathcal{F}(T)) = \mathbb{P}(X(T+t) \in \Gamma \mid X(T)) = g(X(T)) \quad \text{holds } \mathbb{P} - \text{a.e. on } \{T < \infty\}.$$

Here  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  is a bounded measurable function that may depend on  $t$  and  $\Gamma$ , but not on  $T$ .  $\square$

It is clear that every strong MARKOV process with a one-parameter transition semigroup is time-homogeneous strongly Markovian. Also, a diffusion, or a strong MARKOV family, is time-homogeneous strongly Markovian under every probability measure in the family (see Definition 5.1, Chapter IV of IKEDA & WATANABE (1989), and Definition 2.6.3 in KARATZAS & SHREVE (1991)). We show here that every solution to a well-posed local martingale problem associated with the triple  $(\sigma, b, \mu)$  induces a time-homogeneous strongly Markovian process. This is an extension of Theorem 5.4.20 in KARATZAS & SHREVE (1991) in the context of subsection 6.2. Its proof given here is in the same context.

**Proposition 9.1.** *Suppose that the local martingale problem associated with the triple  $(\sigma, b, \mu)$  is well-posed, and let  $\mathbb{Q}^x$  be its solution with  $\omega_2(0) = x$ ,  $\mathbb{Q}^x$ -a.e. Then for every stopping time  $T$  of  $\mathbb{F}_2$ ,  $C \in \mathcal{F}_2$ , and  $x \in \mathbb{R}^2$ , the process  $\omega_2(\cdot)$  satisfies the property*

$$\mathbb{Q}^x(\theta_T^{-1}C \mid \mathcal{F}_2(T))(\omega_2) = \mathbb{Q}^{\omega_2(T)}(C), \quad \mathbb{Q}^x - \text{a.e. on } \{T < \infty\},$$

where  $\theta_T$  is the shift operator  $(\theta_T(\omega_2))(\cdot) := \omega_2(T(\omega_2) + \cdot)$ . In particular,  $\omega_2(\cdot)$  is time-homogeneous strongly Markovian with respect to  $(\Omega_2, \mathcal{F}_2, \mathbb{Q}^x)$  and the filtration  $\mathbb{F}_2$ , for every  $x \in \mathbb{R}^2$ .

We shall need a countable determining class for our local martingale problem, so we introduce it next. A crucial result in this regard, Lemma 9.1 below, is proved in an Appendix, section 11.

**Definition 9.2.** We shall denote by  $\mathfrak{E} \subseteq \mathfrak{D}_+^\mu$  the collection that consists of

- (i) the functions  $g_A(x) := \|x\| (\mathbf{1}_A(\arg(x)) - \nu(A))$  as in (6.22), where  $A \subset [0, 2\pi)$  is of the form  $[a, b)$  and  $a, b$  are rational numbers; and of
- (ii) the following functions in  $\mathfrak{D}_+^\mu$  used in the proof of Part (b) of Proposition 6.1: namely,  $g_1, g_2, g_{i,k}, 1 \leq i, k \leq 2$  in (6.12);  $g_{1,1}^\circ, g_{2,2}^\circ, g_3$  in (6.13); as well as, for every rational  $c_1 > 0$ , a function  $g_4 \in \mathfrak{D}^\mu$  of the form  $g_4(r, \theta) = \psi(r)$  where  $\psi : [0, \infty) \rightarrow [0, \infty)$  is smooth with  $\psi(r) = r$  for  $r \geq c_1$ .

In particular,  $\mathfrak{E}$  is a *countable collection*.  $\square$

**Lemma 9.1.** Suppose  $\mathbb{Q}$  is a probability measure on  $(\Omega_2, \mathcal{F}_2)$  with  $\omega_2(0) = x$ ,  $\mathbb{Q}$ -a.e., under which  $M^g(\cdot; \omega_2)$  is a continuous local martingale (resp., submartingale) of the filtration  $\mathbb{F}_2$  for every function  $g \in \mathfrak{D}^\mu \cap \mathfrak{E}$  (resp.,  $\mathfrak{E}$ ). Then this is also true for every function  $g \in \mathfrak{D}^\mu$  (resp.,  $\mathfrak{D}_+^\mu$ ).

*Proof of Proposition 9.1:* We proceed as in KARATZAS & SHREVE (1991), proof of Theorem 5.4.20, including Lemma 5.4.18 and Lemma 5.4.19. It is easy to check that all the arguments there apply to our context (with some standard localization and application of optional sampling to submartingales), except for the final step of the proof of Lemma 5.4.19. To get through it, we only need to find a countable collection  $\mathfrak{E} \subset \mathfrak{D}_+^\mu$  with the property that, in order to show that  $M^g(\cdot; \omega_2)$  is a continuous local martingale (resp., submartingale) for every function  $g \in \mathfrak{D}^\mu$  (resp.,  $\mathfrak{D}_+^\mu$ ), it suffices to have these properties for all functions in  $\mathfrak{E}$ . We appeal now to Lemma 9.1, and the proof of Proposition 9.1 follows.  $\square$

## 9.2 Time-homogeneous Strongly Markovian Solutions to (6.4), under only (2.17)

Let us recall Part (a) of Proposition 6.1. Suppose that we do not specify a measure  $\mu$  in advance, and that the “thinning” condition (2.18) is not imposed. In particular, with given BOREL-measurable functions  $b : [0, \infty) \rightarrow \mathbb{R}$ ,  $\sigma : [0, \infty) \rightarrow \mathbb{R} \setminus \{0\}$  and real numbers  $\gamma_i, i = 1, 2$ , we consider the system of stochastic equations (6.4) subject only to the non-stickiness condition (2.17).

From Part (b) of Proposition 6.1 we know that, for a probability measure  $\mu$  on  $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$  with

$$\gamma_i = \int_{\mathfrak{S}} \mathfrak{f}_i(z) \mu(dz), \quad i = 1, 2, \quad (9.1)$$

every solution to the local martingale problem associated with the triple  $(\sigma, b, \mu)$  induces a solution to the system (6.4), subject to (2.17). But can we obtain all the solutions of (6.4), (2.17) in this way?

The answer is negative: There are usually several probability measures  $\mu$  satisfying (9.1), so we can construct a solution to (6.4) that satisfies (2.17) and features two different “spinning measures”, both satisfying (9.1). Then this solution is not related to that of a local martingale problem associated with the triple  $(\sigma, b, \mu)$ , for any  $\mu$ . The construction will be given in detail at the end of this subsection (Remark 9.4).

Interestingly, if we restrict our scope to solutions with some appropriate time-homogeneous strong MARKOV properties, then each solution to (6.4) subject to the non-stickiness condition (2.17) is related to that of a local martingale problem associated with the triple  $(\sigma, b, \mu)$ , for some  $\mu$  that depends on this solution. This is the main result of the present subsection, Theorem 9.1 below.

Before stating this result, we note that Proposition 2.6.6 (c') in KARATZAS & SHREVE (1991), proved for strong MARKOV families, admits a version for continuous, time-homogeneous strongly Markovian processes with exactly the same proof. We state this version here; it will be used several times in what follows.

**Proposition 9.2.** Suppose  $X(\cdot)$  is continuous and time-homogeneous strongly Markovian, in the sense of Definition 9.1. Then for every set  $B \in \mathcal{B}(C[0, \infty)^d)$  we have

$$\mathbb{P}(X(T + \cdot) \in B \mid \mathcal{F}(T)) = \mathbb{P}(X(T + \cdot) \in B \mid X(T)) = \mathfrak{h}(X(T)), \quad \mathbb{P} - \text{a.e. on } \{T < \infty\}.$$

for some bounded, measurable function  $\mathfrak{h} : \mathbb{R}^d \rightarrow \mathbb{R}$  that may depend on the set  $B$ , but not on  $T$ .

We present now the main result of this section. Its proof is given in an Appendix, section 12.

**Theorem 9.1. Strongly Markovian Solutions of (6.4), (2.17):** *Let us consider a weak solution  $(X(\cdot), W(\cdot))$ ,  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$  of the system (6.4) for some given constants  $\gamma_1, \gamma_2$ , namely*

$$X_i(\cdot) = X_i(0) + \int_0^\cdot \mathfrak{f}_i(X(t)) \left[ \mathbf{b}(\|X(t)\|) dt + \boldsymbol{\sigma}(\|X(t)\|) dW(t) \right] + \gamma_i L^{\|X\|}(\cdot), \quad i = 1, 2.$$

(i) *The radial part  $\|X(\cdot)\|$  of the state process solves then the equation (6.5), namely*

$$\|X(\cdot)\| = \|X(0)\| + \int_0^\cdot \mathbf{1}_{\{\|X(t)\| > 0\}} \left( \mathbf{b}(\|X(t)\|) dt + \boldsymbol{\sigma}(\|X(t)\|) dW(t) \right) + L^{\|X\|}(\cdot).$$

(ii) *If both  $X(\cdot)$  and  $\|X(\cdot)\|$  are time-homogeneous, strongly Markovian processes with respect to  $\mathbb{F}^X = \{\mathcal{F}^X(t)\}_{0 \leq t < \infty}$  and (2.17) holds, then there exists a probability measure  $\boldsymbol{\mu}$  on  $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$  such that  $X(\cdot)$  induces a solution to the local martingale problem associated with the triple  $(\boldsymbol{\sigma}, \mathbf{b}, \boldsymbol{\mu})$ .*

(iii) *If, in addition, the state process of this weak solution satisfies the analogue*

$$\mathbb{P}(L^{\|X\|}(\infty) > 0) > 0 \quad (9.2)$$

*of the condition (3.6), then the measure  $\boldsymbol{\mu}$  in (ii) is uniquely determined by  $X(\cdot)$  and must satisfy (9.1).*

**Remark 9.1.** The existence and determination of  $\boldsymbol{\mu}$  in Theorem 9.1(ii)(iii) are reminiscent of what happens for the skew BM, where one can “read off” from the HARRISON & SHEPP (1981) equation – e.g. (10.2) below – what the skewness parameter is. The measure  $\boldsymbol{\mu}$  here, however, cannot be decided only through equation (6.4), as one can usually find many  $\boldsymbol{\mu}$ ’s satisfying (9.1), given  $\gamma_1, \gamma_2$ . Rather,  $\boldsymbol{\mu}$  can be observed from the paths of a given solution, as (12.1) and Proposition 12.1 will show in the proof of Theorem 9.1.

Under appropriate conditions on  $(\boldsymbol{\sigma}, \mathbf{b})$  in (6.4), we will only need  $X(\cdot)$  itself to be time-homogeneous and strongly Markovian with respect to  $\mathbb{F}^X$  in Theorem 9.1(ii). The following lemma guarantees this.

**Lemma 9.2.** *With the setting and assumptions of Theorem 9.1, suppose that the local submartingale problem of subsection 6.1 associated with the pair  $(\boldsymbol{\sigma}, \mathbf{b})$  is well-posed.*

*Then  $\|X(\cdot)\|$  is time-homogeneous strongly Markovian with respect to  $\mathbb{F}^X$ .*

**Proof of Lemma 9.2:** We obtain the equation (6.5) for the radial part  $\|X(\cdot)\|$  from Theorem 9.1(i). Applying Itô’s formula to it in the context of subsection 6.1, we see that for every function  $\psi \in C^2([0, \infty); \mathbb{R})$  with  $\psi'(0+) \geq 0$  (resp.  $\psi'(0+) = 0$ ), the process  $K^\psi(\cdot; \|X(\cdot)\|)$  is a continuous local submartingale (resp. martingale) with respect to the filtration  $\mathbb{F}$ . This process is also adapted to  $\mathbb{F}^X$  and  $\mathcal{F}^X(t) \subseteq \mathcal{F}(t)$  holds for all  $t \geq 0$ , so the statement in the last sentence still holds with  $\mathbb{F}$  replaced by  $\mathbb{F}^X$ .

Following the idea of Lemma 5.4.18 and Lemma 5.4.19 in KARATZAS & SHREVE (1991), we denote by  $\mathbb{Q}_\omega(A) = \mathbb{Q}(\omega; A) : \Omega \times \mathcal{F} \mapsto [0, 1]$  the regular conditional probability for  $\mathcal{F}$  given  $\mathcal{F}^X(T)$ , where  $T$  is a bounded stopping time of  $\mathbb{F}^X$ . For every  $\omega \in \Omega$ , we define the probability measure  $\mathbb{P}_\omega$  on  $(C[0, \infty), \mathcal{B}(C[0, \infty)))$  via  $\mathbb{P}_\omega(F) := \mathbb{Q}_\omega(\|X(T + \cdot)\| \in F)$ ,  $\forall F \in \mathcal{B}(C[0, \infty))$ .

With this notation and the conclusion in the first paragraph of this proof, we can follow the arguments in the aforementioned two lemmas to show that for a.e.  $\omega \in \Omega$ , the probability measure  $\mathbb{P}_\omega$  solves the local submartingale problem associated with the pair  $(\boldsymbol{\sigma}, \mathbf{b})$ , starting at  $\|X(T, \omega)\|$ . Combining this with the well-posedness of the local submartingale problem, we prove Lemma 9.2 by applying the proof of Theorem 5.4.20 in KARATZAS & SHREVE (1991).  $\square$

**Remark 9.2.** Just as in the proof of Proposition 9.1, the above argument needs a “countable representatives” result like Lemma 9.1. Here it suffices to take functions of the form  $f(x) = x$ ,  $g(x) = x^2$ , and for every  $n \in \mathbb{N}$  a function  $f_n(\cdot)$  such that  $f_n'(0+) = 0$  and  $f_n(x) = x$  for  $x \geq (1/n)$ .

In conjunction with Lemma 9.2, Theorem 9.1 has the following corollary.

**Corollary 9.1.** *Suppose that the conditions (2.17), (9.2) are satisfied by a weak solution  $(X(\cdot), W(\cdot))$ ,  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$  of the system of equations (6.4), for some given real numbers  $\gamma_1, \gamma_2$ . Suppose also that the local submartingale problem associated with the pair  $(\sigma, \mathbf{b})$  is well-posed.*

*If the state process  $X(\cdot)$  of this weak solution is time-homogeneous and strongly Markovian with respect to  $\mathbb{F}^X$ , then it determines a probability measure  $\mu$  on  $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$  which satisfies (9.1), and is such that  $X(\cdot)$  induces a solution to the local martingale problem associated with the triple  $(\sigma, \mathbf{b}, \mu)$ .*

### 9.3 The case of WALSH Brownian Motion

Let us specialize now the system of equations (6.4) to the case  $\mathbf{b} = \mathbf{0}$ ,  $\sigma = \mathbf{1}$  as in Proposition 7.2, namely

$$X_i(\cdot) = x_i + \int_0^\cdot f_i(X(t)) dW(t) + \gamma_i L^{\|X\|}(\cdot), \quad i = 1, 2. \quad (9.3)$$

We shall show that when  $\gamma_1^2 + \gamma_2^2 \leq 1$  this system, coupled with the time-homogeneous strong MARKOV property, characterizes WALSH Brownian motions under the non-stickiness condition (2.17). We note that in the statement and proof of the next proposition, neither  $(\gamma_1, \gamma_2)$  nor  $\mu$  are specified in advance; rather, we view them as related via (9.1).

**Proposition 9.3. A New Characterization of WALSH Brownian Motions :** *Assume that  $Z(\cdot)$  is a continuous, adapted planar process on some filtered probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ ,  $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}(t)\}_{0 \leq t < \infty}$ . Then the following two assertions are equivalent:*

- (i)  *$Z(\cdot)$  is a WALSH Brownian motion, defined via the semigroup (7.1), for some spinning measure  $\mu$ .*
- (ii) *For some pair of real numbers  $(\gamma_1, \gamma_2)$  with  $\gamma_1^2 + \gamma_2^2 \leq 1$ , there exists a weak solution  $(X(\cdot), W(\cdot))$ ,  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$  to the system of equations (9.3), such that  $X(\cdot)$  : is time-homogeneous strongly Markovian with respect to  $\mathbb{F}^X$ ; satisfies the condition (2.17); and has the same distribution as  $Z(\cdot)$ .*

*When these assertions hold, the measure  $\mu$  of the statement (i), and the coefficients  $\gamma_1, \gamma_2$  of the statement (ii), satisfy the relationship (9.1).*

*Proof:* (i)  $\Rightarrow$  (ii). On the strength of Propositions 7.1 and 6.1, the process  $Z(\cdot)$  induces a weak solution of (9.3) subject to (2.17), where  $\gamma_1, \gamma_2$  are given by (9.1) and therefore satisfy  $\gamma_1^2 + \gamma_2^2 \leq 1$ . Since  $Z(\cdot)$  is time-homogeneous strongly Markovian with respect its own filtration, so is this solution.

(ii)  $\Rightarrow$  (i). Appealing to Proposition 6.2, we see that the local submartingale problem associated with the pair  $(\mathbf{1}, \mathbf{0})$  is well-posed. By Theorem 9.1(i) we obtain that the radial part of  $X(\cdot)$  satisfies

$$\|X(\cdot)\| = \|X(0)\| + \int_0^\cdot \mathbf{1}_{\{\|X(t)\| > 0\}} dW(t) + L^{\|X\|}(t) = \|X(0)\| + W(\cdot) + L^{\|X\|}(\cdot), \quad (9.4)$$

with the help of the non-stickiness condition (2.17). Therefore,  $\|X(\cdot)\|$  is the SKOROKHOD reflection of the Brownian motion  $\|X(0)\| + W(\cdot)$ , and satisfies  $\mathbb{P}(L^{\|X\|}(\infty) = \infty) = 1$ , so the condition (9.2) follows.

Now from Corollary 9.1, the weak solution posited in (ii) induces a solution to the local martingale problem associated with the triple  $(\mathbf{1}, \mathbf{0}, \mu)$ , for some probability measure  $\mu$  that satisfies (9.1). Propositions 7.1 and 6.2 show that  $X(\cdot)$  is a WALSH Brownian motion with spinning measure  $\mu$ , and so is  $Z(\cdot)$ .  $\square$

**Remark 9.3. Similarities and Differences:** Proposition 9.3 and Theorem 3.1 show that the system of equations (9.3), with the condition  $\gamma_1^2 + \gamma_2^2 \leq 1$  on the coefficients, is a two-dimensional analogue of the HARRISON & SHEPP (1981) equation for the skew Brownian motion. But with the following caveat:

The equations (9.3), (2.17) characterize WALSH Brownian motions only when we restrict attention to time-homogeneous strongly Markovian processes. *If this restriction is not imposed, there will be solutions to the system (9.3) that are not WALSH Brownian motions.* Such solutions are discussed in the next remark.

Furthermore, (9.3) does not describe a unique WALSH Brownian motion, but may be satisfied by many such motions with different spinning measures (cf. Remark 9.1). By contrast, we can read off the flipping probability from the coefficient in the equation for the one-dimensional skew Brownian motion. The construction in Remark 9.4 right below is actually based on this observation.

*Remark 9.4. A solution to the system of equations (9.3) that features two different spinning measures:* Consider the system of equations (9.3) with  $\gamma_1 = \gamma_2 = 0$  and  $x = (0, 0)$ , and note that both measures

$$\mu_1 = \frac{1}{2} \delta_{(1,0)} + \frac{1}{2} \delta_{(-1,0)} \quad \text{and} \quad \mu_2 = \frac{1}{2} \delta_{(0,1)} + \frac{1}{2} \delta_{(0,-1)}$$

satisfy (9.1). Let  $X(\cdot)$  be a WALSH Brownian motion that solves the system (9.3) with  $X(0) = (0, 0)$ ,  $\gamma_1 = \gamma_2 = 0$ , spinning measure  $\mu_1$  and driving Brownian motion  $B(\cdot)$ . Let  $Y(\cdot)$  be another WALSH Brownian motion that solves (9.3) with  $Y(0) = (1, 0)$ ,  $\gamma_1 = \gamma_2 = 0$ , spinning measure  $\mu_2$  and driver  $\tilde{B}(\cdot) := B(\tau_{(1,0)} + \cdot)$ , another Brownian motion. Now define  $\tau_{(1,0)} := \inf\{t \geq 0 : X(t) = (1, 0)\}$  and

$$Z(t) := X(t), \quad 0 \leq t < \tau_{(1,0)}, \quad \text{and} \quad Z(\tau_{(1,0)} + t) := Y(t), \quad \forall t \geq 0.$$

The so-defined process  $Z(\cdot)$  solves (9.3) with  $Z(0) = (0, 0)$ ,  $\gamma_1 = \gamma_2 = 0$  and driving Brownian motion  $B(\cdot)$ , but is *not* a WALSH Brownian motion: it switches from  $\mu_1$  to  $\mu_2$  after time  $\tau_{(1,0)}$ . It is also not time-homogeneous strongly Markovian, by virtue of either Proposition 9.3 or elementary observations.

## 10 Examples

**Example 10.1. WALSH's Brownian Motion and Spider Martingales:** When the spinning measure  $\mu$  in Theorem 2.1 is a discrete probability charging a finite number of rays that pass through the origin, and the driving semimartingale  $U(\cdot)$  is Brownian motion, the process  $X(\cdot)$  becomes the original WALSH Brownian motion  $W(\cdot)$  with roundhouse singularities in multipole fields as in Proposition 7.2. Given a finite number  $m \geq 2$  of distinct angles  $\{\theta_\ell \in [0, 2\pi), \ell = 1, \dots, m\}$ , let us consider  $m$  rays emanating from the origin,

$$\mathcal{I}_\ell := \{x \in \mathbb{R}^2 \setminus \{0\} : \arg(x) = \theta_\ell\}, \quad \ell = 1, \dots, m,$$

and assign a discrete probability measure  $\mu$  with weights  $p_\ell \in (0, 1)$ ,  $\sum_{\ell=1}^m p_\ell = 1$ , such that

$$\mu(\{(\cos(\theta_\ell), \sin(\theta_\ell))\}) = \mathbb{P}(\arg(\xi_1) = \theta_\ell) = p_\ell, \quad \ell = 1, \dots, m. \quad (10.1)$$

Using MARKOV semigroups and excursions, BARLOW, PITMAN & YOR (1989) study WALSH's Brownian motion  $W(\cdot)$  on the collection of rays  $\bigcup_{\ell=1}^m \mathcal{I}_\ell \cup \{0\}$ . Their approach has been generalized to “multiple spider martingales” by YOR (1997), and has been studied by TSIREL'SON (1997), BARLOW, ÉMERY, KNIGHT, SONG & YOR (1998), WATANABE (1999) and MANSUY & YOR (2006), pp. 103-116.

**Example 10.2. The Case of Two Rays:** Let us consider the setup of the previous example with  $m = 2$  and  $\theta_1 = 0$ ,  $\theta_2 = \theta \in (0, \pi]$ , as well as  $\mathbb{P}(\arg(\xi_1) = \theta) = p \in (0, 1)$ . The equations of (3.4) become

$$\begin{aligned} X_1(\cdot) &= x_1 + \int_0^\cdot \cos(\arg(X(t))) dU(t) + (1 - p + p \cos(\theta)) L^{\|X\|}(\cdot), \\ X_2(\cdot) &= x_2 + \int_0^\cdot \sin(\arg(X(t))) dU(t) + p \sin(\theta) L^{\|X\|}(\cdot) \end{aligned}$$

with  $L^{X_1}(\cdot) = (1 - p + p \cos^+(\theta)) L^{\|X\|}(\cdot)$  and  $L^{X_2}(\cdot) = p \sin^+(\theta) L^{\|X\|}(\cdot)$ .

**Case I:** With  $\theta = \pi$ , and with  $x_2 = 0$  for simplicity, the second of these equations has the trivial solution  $X_2(\cdot) \equiv 0$ , whereas the first can be cast in the form of the celebrated HARRISON & SHEPP (1981) equation

$$X_1(\cdot) = x_1 + V_1(\cdot) + \frac{1-2p}{1-p} L^{X_1}(\cdot), \quad \text{driven by} \quad V_1(\cdot) := \int_0^\cdot \operatorname{sgn}(X_1(t)) dU(t). \quad (10.2)$$

As these authors showed, when  $U(\cdot)$  is Brownian motion the above equation has a pathwise unique, strong solution with respect to the Brownian motion  $V_1(\cdot)$ , and in this case  $X_1(\cdot)$  is skew Brownian motion.

When written in terms of the original driver  $U(\cdot)$ , the above equation for  $X_1(\cdot)$  is a skew version of the TANAKA equation. In particular, Proposition 2.1 of ICHIBA & KARATZAS (2014) establishes the filtration comparisons  $\mathbb{F}^U(\cdot) \subsetneq \mathbb{F}^{V_1}(\cdot) = \mathbb{F}^{X_1}(\cdot) = \mathbb{F}^{(X_1, X_2)}(\cdot)$  when  $U(\cdot)$  is Brownian motion.

**Case II:** When  $\theta \in (0, \pi)$ , we assume for simplicity  $\arg(x) \in \{0, \theta\}$  and consider the process

$$\Upsilon(\cdot) := \frac{-X_2(\cdot)}{\sin(\theta)} \cdot \mathbf{1}_{\{X_2(\cdot) > 0\}} + \frac{X_1(\cdot)}{\cos(\theta)} \cdot \mathbf{1}_{\{X_2(\cdot) = 0\}} :$$

we flatten the state space by rotating the ray. This process also satisfies a HARRISON-SHEPP-type equation

$$\Upsilon(\cdot) = \Upsilon(0) + V_\bullet(\cdot) + \frac{1-2p}{1-p} L^\Upsilon(\cdot) \quad \text{driven by} \quad V_\bullet(\cdot) := \int_0^\cdot \operatorname{sgn}(\Upsilon(t)) dU(\cdot);$$

and conversely, the coördinate processes are given in terms of  $\Upsilon(\cdot)$  as

$$X_1(\cdot) = \Upsilon(\cdot) \cdot \mathbf{1}_{\{\Upsilon(\cdot) > 0\}} - \Upsilon(\cdot) \cos(\theta) \cdot \mathbf{1}_{\{\Upsilon(\cdot) \leq 0\}}, \quad X_2(\cdot) = -\Upsilon(\cdot) \sin(\theta) \cdot \mathbf{1}_{\{\Upsilon(\cdot) \leq 0\}}.$$

If  $U(\cdot)$  is standard Brownian motion, then so is  $V_\bullet(\cdot)$ ; in this case  $\Upsilon(\cdot)$  becomes a skew Brownian motion, and we obtain as before the filtration comparisons  $\mathbb{F}^U(\cdot) \subsetneq \mathbb{F}^{V_\bullet}(\cdot) = \mathbb{F}^\Upsilon(\cdot) = \mathbb{F}^{(X_1, X_2)}(\cdot)$ .

- We have shown that the filtration  $\mathbb{F}^{(X_1, X_2)}(\cdot)$  of a WALSH Brownian motion on two rays coincides with the filtration generated by some standard Brownian motion, and is *strictly finer* than the filtration  $\mathbb{F}^U(\cdot)$  generated by its driving Brownian motion.

- Suppose the driver  $U(\cdot)$  is a continuous local martingale with  $U(0) = 0$  and  $\langle U \rangle(\infty) = \infty$ , and consider its DAMBIS-DUBINS-SCHWARZ representation  $U(\cdot) = \beta(\langle U \rangle(\cdot))$  with  $\beta(\cdot)$  a standard Brownian motion.

From the above considerations and in conjunction with Proposition 2.2 in ICHIBA & KARATZAS (2014) we see that, in the case of a spinning measure  $\mu$  that charges exactly two points on the unit circumference, uniqueness in distribution holds for the system (3.5) subject to (2.13) and (2.18), provided that either

- (i)  $U(\cdot)$  is *pure* (i.e.,  $\langle U \rangle(t)$  is  $\mathcal{F}^\beta(\infty)$ -measurable, for every  $t \in [0, \infty)$ ); or that
- (ii) the quadratic variation process  $\langle U \rangle(\cdot)$  is adapted to a Brownian motion  $\Gamma(\cdot) = (\Gamma_1(\cdot), \dots, \Gamma_n(\cdot))'$  with values in some Euclidean space and independent of the Brownian motion  $\beta(\cdot)$ .

**Example 10.3. TSIREL'SON'S triple point:** When  $\alpha_i^{(+)} = \alpha_i^{(-)}$  for  $i = 1, 2$ , the equations (2.12) and (3.4) become, respectively,

$$X_i(T) = x_i + \int_0^T f_i(X(t)) dS(t) \quad \text{and} \quad X_i(T) = x_i + \int_0^T f_i(X(t)) dU(t); \quad i = 1, 2.$$

This is the case when the common probability distribution  $\mu$  of the I.I.D. random variables  $\{\xi_1, \xi_2, \dots\}$  in (5.1) has zero expectation, namely  $\mathbb{E}[\xi_1] = \mathbf{0}$ . For instance, when  $\mu$  assigns equal weights of  $1/3$  to three points at angles  $\theta_0 + (2\pi\ell/3)$ ,  $\ell = 0, 1, 2$  on the unit circumference  $\mathfrak{S}$  that trisect it, namely,

$$\mathbb{P}(\xi_1 = (\cos(\theta_0 + (2\pi\ell/3)), \sin(\theta_0 + (2\pi\ell/3)))' ) = 1/3; \quad \ell = 0, 1, 2$$

for some  $\theta_0 \in [0, 2\pi)$ . If, in addition,  $U(\cdot) = W(\cdot)$  is Brownian motion, and thus the SKOROKHOD reflection  $S(\cdot) = W(\cdot) + \max_{0 \leq s \leq \cdot} (-W(s))^+$  in (2.2) is a reflecting Brownian motion, we deduce from subsection 3.2 that the corresponding planar process  $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$  is a martingale, to wit  $X_i(T) = x_i + \int_0^T f_i(X(t)) dW(t)$ ,  $i = 1, 2$ .

It was conjectured by BARLOW, PITMAN & YOR (1989), and shown in the landmark paper by TSIREL'SON (1997) (cf. YOR (1997), MANSUY & YOR (2006)), that the natural filtration of this martingale  $X(\cdot)$  is not generated by *any* Brownian motion of *any* dimension.



**Example 10.4. WALSH's Brownian motion with polar drifts:** Let us look at the case  $\sigma(\cdot) \equiv 1$  and  $c(\cdot) \equiv -\lambda$  for some  $\lambda > 0$  in Corollary 6.3. The driving one-dimensional semimartingale  $U(\cdot)$  for  $X(\cdot)$  is Brownian motion with negative drift  $-\lambda$  and with instantaneous reflection at the origin. It follows from Theorem 2.1 the process  $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$  satisfies

$$X_i(T) = x_i + \int_0^T f_i(X(t))(-\lambda dt + dW(t)) + \gamma_i L^{\|X\|}(T), \quad 0 \leq T < \infty$$

for  $i = 1, 2$ , where  $W(\cdot)$  is one-dimensional standard Brownian motion.

Moreover, following Proposition 8.1, we may replace the constant drifts by *drifts exhibiting angular dependence*. Suppose that  $a(r, \theta) = 1$  and  $b(r, \theta) = \lambda(\theta)$  for some measurable function  $\lambda : [0, 2\pi) \rightarrow (0, \infty)$ . The resulting process  $Y(\cdot)$  in Proposition 8.1 has the dynamics

$$Y(T) = y + \int_0^T f(Y(t))(-\lambda(\arg(Y(t)))dt + dW(t)) + \gamma L^{\|Y\|}(T), \quad 0 \leq T < \infty.$$

Since the driving semimartingale is positive recurrent in  $\mathbb{R}_+$ , the degenerate planar process  $X(\cdot)$  is positive recurrent. Its stationary distribution is expressed in polar coördinates as

$$\left( \int_0^{2\pi} \frac{\nu(du)}{2[\lambda(u)]^2} \right)^{-1} \frac{e^{-2\lambda(\theta)r}}{\lambda(\theta)} dr \nu(d\theta); \quad r > 0, \theta \in [0, 2\pi)$$

by the distribution of occupation times and the excursion theory of SALMINEN, VALLOIS & YOR (2007). If  $\lambda(\cdot) \equiv \lambda$  (constant), then the stationary distribution reduces to  $(2\lambda e^{-2\lambda r} dr) \nu(d\theta)$ ,  $r > 0$ ,  $\theta \in [0, 2\pi)$ .

**Example 10.5. WALSH semimartingale driven by BESSEL processes:** Suppose that  $R^2(\cdot)$  is a squared BESSEL process with dynamics  $dR^2(t) = \delta dt + 2\sqrt{R^2(t)} dW(t)$ , where  $\delta \in (1, 2)$  and  $W(\cdot)$  is one-dimensional standard Brownian.

We take the square root  $|R(\cdot)|$  of this process as the driving semimartingale, i.e.,  $U(\cdot) = |R(\cdot)| = S(\cdot)$  in Theorem 2.1. This process  $S(\cdot)$  does not accumulate local time at the origin, i.e.,  $L^S(\cdot) \equiv 0$  holds for  $\delta \in (1, 2)$ , hence the resulting planar process  $X(\cdot)$  of Theorem 2.1 has the dynamics

$$X_i(T) = x_i + \int_0^T f_i(X(t)) \left( \frac{\delta - 1}{2\|X(t)\|} \cdot \mathbf{1}_{\{\|X(t)\| \neq 0\}} dt + dW(t) \right), \quad 0 \leq T < \infty$$

for  $i = 1, 2$ . Note that when  $\delta = 1$ , the process  $X(\cdot)$  becomes WALSH Brownian motion; when  $\delta \in (0, 1)$ , the semimartingale property is violated; when  $\delta \geq 2$ , the process  $R(\cdot)$  never reaches the origin.

Furthermore, and by analogy with Example 10.4, given a measurable function  $\delta : [0, 2\pi) \rightarrow (1, 2)$  we may use the time-change technique with the dispersion  $a(r, \theta) = 4r$  and the drift  $b(r, \theta) = \delta(\theta)$  and consider the WALSH semimartingale  $Y(\cdot)$  driven by angular dependent, squared-BESSEL process

$$Y(T) = y + \int_0^T f(Y(t)) \left( \delta(\arg(Y(t))) dt + 2\sqrt{\|Y(t)\|} dW(t) \right), \quad 0 \leq T < \infty.$$

Here, the process  $\|Y(\cdot)\|$  does not accumulate local time at the origin. The corresponding scale function, inverse function and stochastic clock are given by  $p_\theta(r) = r^{(2-\delta(\theta))/2}$ ,  $q_\theta(r) = r^{2/(2-\delta(\theta))}$ , and

$$\mathcal{T}(\cdot) = \int_0^\cdot \left( (2 - \delta(\theta))^2 r^{-(\delta(\theta)-1)} \right) \Big|_{r=\|Y(t)\|, \theta=\arg(Y(t))} dt,$$

respectively. It can be shown that the stochastic clock does not explode (cf. Lemma 3.1 of BIANE & YOR (1987), Proposition XI.1.11 of REVUZ & YOR (1999), pages 285-289 of ROGERS & WILLIAMS (2000) and Appendix A.1 of ICHIBA ET AL. (2011)). From this process  $Y(\cdot)$  we may define now the WALSH semimartingale  $\Xi(\cdot) = (\Xi_1(\cdot), \Xi_2(\cdot))'$  with  $\Xi_i(\cdot) := f_i(Y(\cdot)) \|Y(\cdot)\|^{1/2}$ ,  $i = 1, 2$  driven by a BESSEL process with angular dependence, which satisfies the vector integral equation derived from (8.12), namely,

$$\Xi(T) = \Xi(0) + \int_0^T \mathfrak{f}(\Xi(t)) \left( \frac{\delta(\arg(\Xi(t))) - 1}{2 \|\Xi(t)\|} \mathbf{1}_{\{\|\Xi(t)\| \neq 0\}} dt + dW(t) \right), \quad 0 \leq T < \infty.$$

## 11 Appendix: The Proof of Lemma 9.1

We denote by  $\widetilde{\mathfrak{D}}^\mu$  (resp.  $\widetilde{\mathfrak{D}}_+^\mu$ ) the collection of functions  $g$  in  $\mathfrak{D}^\mu$  (resp.  $\mathfrak{D}_+^\mu$ ) such that  $M^g(\cdot; \omega_2)$  is a continuous local martingale (resp. submartingale) of the filtration  $\mathbb{F}_2$ , under  $\mathbb{Q}$ . Then we have  $\widetilde{\mathfrak{D}}^\mu \supseteq \mathfrak{D}^\mu \cap \mathfrak{E}$  and  $\widetilde{\mathfrak{D}}_+^\mu \supseteq \mathfrak{E}$  by assumption. The goal here is to show  $\widetilde{\mathfrak{D}}^\mu = \mathfrak{D}^\mu$  and  $\widetilde{\mathfrak{D}}_+^\mu = \mathfrak{D}_+^\mu$ .

Recalling that  $\mathfrak{E}$  contains the functions in Definition 9.2(ii), we can follow the proof of Part (b) of Proposition 6.1 and show that there exists a one-dimensional standard Brownian motion  $W(\cdot)$  on an extension of the filtered probability space  $(\Omega_2, \mathcal{F}_2, \mathbb{Q})$ ,  $\mathbb{F}_2$  such that (6.4), (6.5) hold with  $X(\cdot)$  given by (6.9), or simply  $X(\cdot) := \omega_2(\cdot)$ . It is clear, therefore, that  $\int_0^t \mathbf{1}_{\{\|\omega_2(u)\| > 0\}} (|\mathbf{b}(\|\omega_2(u)\|)| + \mathbf{a}(\|\omega_2(u)\|)) du < \infty$  holds for all  $0 \leq t < \infty$ ,  $\mathbb{Q}$ -a.s. We make now the following two observations.

**First Observation:**  $\widetilde{\mathfrak{D}}^\mu$  is a linear space. This is obvious from the linearity of stochastic integrals, derivatives, and local martingales.

**Second Observation:** Suppose  $\{g_n\}_{n \in \mathbb{N}} \subseteq \widetilde{\mathfrak{D}}^\mu$  and  $g \in \mathfrak{D}^\mu$  satisfy the following: as  $n \uparrow \infty$ ,  $g_n(x) \rightarrow g(x)$ ,  $\forall x \in \mathbb{R}^2$  and  $G'_n(x) \rightarrow G'(x)$ ,  $G''_n(x) \rightarrow G''(x)$ ,  $\forall x \in \mathbb{R}^2 \setminus \{0\}$ , and all these functions  $(g_n, g, G'_n, G', G''_n, G'')$  are uniformly bounded on every compact subset of  $\mathbb{R}^2$ . Then we have  $g \in \widetilde{\mathfrak{D}}^\mu$ .

To see this, we define stopping times

$$T_k = \inf \left\{ t : \int_0^t \mathbf{1}_{\{\|\omega_2(u)\| > 0\}} (|\mathbf{b}(\|\omega_2(u)\|)| + \mathbf{a}(\|\omega_2(u)\|)) du \geq k \text{ or } \|\omega_2(t)\| \geq k \right\}, \quad k \in \mathbb{N},$$

and note that  $M^{g_n}(\cdot \wedge T_k; \omega_2)$ ,  $n \in \mathbb{N}$  are uniformly bounded local martingales, hence uniformly bounded martingales, for all  $k \in \mathbb{N}$ ; and that  $\lim_{n \rightarrow \infty} M^{g_n}(t \wedge T_k; \omega_2) = M^g(t \wedge T_k; \omega_2)$  for any  $t \in [0, \infty)$ . Thus  $M^g(\cdot \wedge T_k; \omega_2)$  is also a continuous martingale, and the conclusion  $g \in \widetilde{\mathfrak{D}}^\mu$  follows.

• Returning to our argument, we know that for the functions of Definition 9.2, the process  $M^{g_A}(\cdot; \omega_2)$  is a local martingale for any interval  $A \subseteq [0, 2\pi)$  of the form  $[a, b)$ , where  $a, b$  are rationals. Thus the same is true when  $A$  is the disjoint union of such intervals, by linearity. These sets form an algebra. By the second observation and monotone class arguments, the same is also true for every BOREL subset  $A$  of  $[0, 2\pi)$ . Now for any two disjoint BOREL subsets  $A, B$  of  $[0, 2\pi)$ , we define

$$g_{A,B}(x) := \|x\| (\nu(A) \mathbf{1}_{\{\arg(x) \in B\}} - \nu(B) \mathbf{1}_{\{\arg(x) \in A\}}) \quad \text{and note} \quad g_{A,B}(x) = \nu(A) g_B(x) - \nu(B) g_A(x),$$

thus  $g_{A,B} \in \widetilde{\mathfrak{D}}^\mu$  by linearity. Starting from this and using linearity and induction, we show that if  $h : [0, 2\pi) \rightarrow \mathbb{R}$  is simple and satisfies  $\int_0^{2\pi} h(\theta) \nu(d\theta) = 0$ , then the mapping  $x \mapsto \|x\| \cdot h(\arg(x))$  is in  $\widetilde{\mathfrak{D}}^\mu$ . Using approximation and the second observation, we see that this statement is still true when “simple” is replaced by “bounded and measurable”.

Let us recall now that, in the second paragraph of this section, we obtained the existence of a one-dimensional standard Brownian motion  $W(\cdot)$  on an extension of the filtered probability space  $(\Omega_2, \mathcal{F}_2, \mathbb{Q})$ ,  $\mathbb{F}_2$ , along with (6.4) and (6.5), where  $X(\cdot) := \omega_2(\cdot)$ . By defining  $S(\cdot) := \|X(\cdot)\|$ , we can follow the proof of Theorem 4.1 to establish for any given function  $g \in \mathfrak{D}^\mu$  with  $g'_\theta(0+) \equiv 0$  the following FREIDLIN-SHEU-type semimartingale decomposition:

$$g(\omega_2(\cdot)) = g(x) + \int_0^\cdot \mathbf{1}_{\{\|\omega_2(t)\| > 0\}} \left( \mathbf{b}(\|\omega_2(t)\|) G'(\omega_2(t)) + \frac{1}{2} \mathbf{a}(\|\omega_2(t)\|) G''(\omega_2(t)) \right) dt$$

$$+ \int_0^\cdot \mathbf{1}_{\{\|\omega_2(t)\| > 0\}} \sigma(\|\omega_2(t)\|) G'(\omega_2(t)) dW(t).$$

The condition (2.18) is *not* needed here; and neither are terms involving local time.

This is because the use of (2.18) in the proof of Theorem 4.1 comes only when proving the convergence to local time as in (5.18). But this property holds here trivially, courtesy of  $g'_\theta(0+) \equiv 0$ . It follows from the above decomposition of FREIDLIN-SHEU-type that, if  $g \in \mathfrak{D}^\mu$  satisfies  $g'_\theta(0+) \equiv 0$ , then  $g \in \widetilde{\mathfrak{D}}^\mu$ .

Finally, we observe that every  $g \in \mathfrak{D}^\mu$  can be decomposed as  $g = g^{(1)} + g^{(2)}$ , where the function  $x \mapsto g^{(1)}(x) := \|x\| \cdot g'_\theta(0+)$  is in  $\widetilde{\mathfrak{D}}^\mu$  by the first paragraph of this bullet, and the function  $g^{(2)} := g - g^{(1)} \in \mathfrak{D}^\mu$  satisfies  $(g^{(2)})'(0+) \equiv 0$ . With the considerations above, we see  $g \in \widetilde{\mathfrak{D}}^\mu$ , thus  $\widetilde{\mathfrak{D}}^\mu = \mathfrak{D}^\mu$ . We decompose then every function  $g \in \mathfrak{D}_+^\mu$  as  $g = g_{(1)} + g_{(2)}$ , where  $g_{(1)}(x) := c\|x\|$  with a constant  $c := \int_0^{2\pi} g'_\theta(0+) \nu(d\theta) \geq 0$  and  $g_{(2)} := g - g_{(1)} \in \mathfrak{D}^\mu$  (cf. Remark 6.1). Here  $M^{g_{(2)}}(\cdot; \omega_2)$  is a local martingale, and  $M^{g_{(1)}}(\cdot; \omega_2) = cM^{g_3}(\cdot; \omega_2)$  is a local submartingale, since the mapping  $x \mapsto g_3(x) = \|x\|$  belongs to  $\mathfrak{E} \subseteq \widetilde{\mathfrak{D}}_+^\mu$  (cf. Definition 9.2 (ii)). Thus  $M^g(\cdot; \omega_2)$  is also a local submartingale and  $g \in \widetilde{\mathfrak{D}}_+^\mu$ . We conclude then  $\widetilde{\mathfrak{D}}_+^\mu = \mathfrak{D}_+^\mu$ , and the proof of Lemma 9.1 is complete.  $\square$

## 12 Appendix: The Proof of Theorem 9.1

We first identify the measure  $\mu$  from  $X(\cdot)$ , using the time-homogeneous strong MARKOV property of this process. Then we establish a FREIDLIN-SHEU-type formula for  $X(\cdot)$ , so as to relate this process to a solution of the local martingale problem associated with the triple  $(\sigma, b, \mu)$ .

- By the result of Part (a) of Proposition 6.1, we obtain the equation (6.5), thus prove Part (i) of Theorem 9.1. We also know that the “direction process”  $\mathfrak{f}(X(\cdot)) = (\mathfrak{f}_1(X(\cdot)), \mathfrak{f}_2(X(\cdot)))$  is constant on every excursion interval of  $\|X(t)\|$ , by applying the idea in the argument at the beginning of section 3.
- For every  $\varepsilon > 0$ , we define the stopping times  $\{\tau_m^\varepsilon, m \in \mathbb{N}_0\}$  as in (5.6).

**PROOF OF THEOREM 9.1(ii), PART A:** Let us start by assuming that, with probability one, all these stopping times  $\{\tau_m^\varepsilon, m \in \mathbb{N}_0\}$  are finite. Then for every  $\varepsilon > 0$ ,  $\ell \in \mathbb{N}_0$ , we define also the measure

$$\mu_\ell^\varepsilon(B) := \mathbb{P}(\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon)) \in B), \quad \forall B \in \mathcal{B}(\mathfrak{S}). \quad (12.1)$$

**Proposition 12.1.** *The measure  $\mu_\ell^\varepsilon$  just introduced does not depend on either  $\varepsilon$  or  $\ell$ , so we can define  $\mu := \mu_\ell^\varepsilon$ ,  $\forall \varepsilon > 0, \ell \in \mathbb{N}_0$ . Furthermore,  $\{\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon))\}_{\ell \in \mathbb{N}_0}$  is a sequence of independent random variables with common distribution  $\mu$ , for every fixed  $\varepsilon > 0$ .*

*Proof: Step 1.* We shall show in this step that  $\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon))$  is independent of  $\mathcal{F}^X(\tau_{2\ell}^\varepsilon)$  for any  $\varepsilon > 0$ ,  $\ell \in \mathbb{N}_0$ , and that the random variables  $\{\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon))\}_{\ell \in \mathbb{N}_0}$  are I.I.D. for any fixed  $\varepsilon > 0$ .

By Proposition 9.2, we have for every  $\varepsilon > 0$ ,  $\ell \in \mathbb{N}_0$ ,  $B \in \mathcal{B}(\mathfrak{S})$ , the identity

$$\mathbb{P}(\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon)) \in B \mid \mathcal{F}^X(\tau_{2\ell}^\varepsilon)) = \mathbb{P}(X(\tau_{2\ell}^\varepsilon + \cdot) \in A_1 \mid \mathcal{F}^X(\tau_{2\ell}^\varepsilon)) = \mathbb{P}(X(\tau_{2\ell}^\varepsilon + \cdot) \in A_1 \mid X(\tau_{2\ell}^\varepsilon)).$$

Here

$$A_1 := \{\omega \in C[0, \infty)^2 : \mathfrak{f}(\omega(\tau_1^\varepsilon(\omega))) \in B, \omega(0) = 0\} \in \mathcal{B}(C[0, \infty)^2),$$

and the above conditional probability also equals  $\mathfrak{h}_1(X(\tau_{2\ell}^\varepsilon))$ , for some bounded measurable function  $\mathfrak{h}_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  that depends only on  $A_1$ . Now because  $X(\tau_{2\ell}^\varepsilon) \equiv 0$ , this conditional probability is a constant that is irrelevant to  $\tau_{2\ell}^\varepsilon$ , in particular, to  $\ell$ . We deduce that  $\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon))$  is independent of  $\mathcal{F}^X(\tau_{2\ell}^\varepsilon)$ , and its distribution does not depend on  $\ell$ . Therefore, the random variables in  $\{\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon))\}_{\ell \in \mathbb{N}_0}$  are I.I.D.

*Step 2:* On the strength of Step 1, we can define  $\mu^\varepsilon := \mu_\ell^\varepsilon$ ,  $\forall \ell \in \mathbb{N}_0$ . We shall show in this step that  $\mu^\varepsilon$  does not depend on  $\varepsilon$ . Once this is done, we shall obtain Proposition 12.1 by combining the results of the two steps. Let  $\varepsilon_1 > \varepsilon_2 > 0$ . We shall prove the claim

$$\mu^{\varepsilon_1}(B) = \mu^{\varepsilon_2}(B), \quad \forall B \in \mathcal{B}(\mathfrak{S}).$$

Since  $\|X(\tau_1^{\varepsilon_1})\| = \varepsilon_1 > \varepsilon_2$ , and  $\|X(\cdot)\| \leq \varepsilon_2$  on every  $[\tau_{2\ell}^{\varepsilon_2}, \tau_{2\ell+1}^{\varepsilon_2}]$ , we see that for a.e.  $\omega \in \Omega$  there exists a unique  $\ell_2 \in \mathbb{N}_0$  (depending on  $\omega$ ), such that  $\tau_{2\ell_2+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell_2+2}^{\varepsilon_2}$ . Then we can partition  $\Omega = \bigcup_{\ell \in \mathbb{N}_0} \{\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}\}$ , where the right-hand side is a disjoint union. On  $\{\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}\}$ , we note that  $\tau_1^{\varepsilon_1}$  and  $\tau_{2\ell+1}^{\varepsilon_2}$  are on the same excursion interval of  $\|X(\cdot)\|$ . Then from the considerations in the first bullet, we have  $f(X(\tau_1^{\varepsilon_1})) = f(X(\tau_{2\ell+1}^{\varepsilon_2}))$  on the event  $\{\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}\}$ .

On the strength of Lemma 12.1 below, we can write

$$\begin{aligned} \mu^{\varepsilon_1}(B) &= \mathbb{P}(f(X(\tau_1^{\varepsilon_1})) \in B) = \sum_{\ell \in \mathbb{N}_0} \mathbb{P}(\{f(X(\tau_1^{\varepsilon_1})) \in B\} \cap \{\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}\}) \\ &= \sum_{\ell \in \mathbb{N}_0} \mathbb{P}(\{f(X(\tau_{2\ell+1}^{\varepsilon_2})) \in B\} \cap \{\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}\} \cap \left\{ \max_{\tau_{2\ell+1}^{\varepsilon_2} \leq t \leq \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \geq \varepsilon_1 \right\}) \\ &= \mu^{\varepsilon_2}(B) \sum_{\ell \in \mathbb{N}_0} \mathbb{P}(\{\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}\} \cap \left\{ \max_{\tau_{2\ell+1}^{\varepsilon_2} \leq t \leq \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \geq \varepsilon_1 \right\}) \\ &= \mu^{\varepsilon_2}(B) \sum_{\ell \in \mathbb{N}_0} \mathbb{P}(\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}) = \mu^{\varepsilon_2}(B). \end{aligned}$$

This way we complete Step 2, and Proposition 12.1 is proved.  $\square$

**Lemma 12.1.** (a) We have the comparisons  $\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}$ , if and only if  $\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}$  and  $\max_{\tau_{2\ell+1}^{\varepsilon_2} \leq t \leq \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \geq \varepsilon_1$  hold.

(b)  $\forall B \in \mathcal{B}(\mathfrak{S})$ , the three events  $\{f(X(\tau_{2\ell+1}^{\varepsilon_2})) \in B\}$ ,  $\{\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}\}$ ,  $\{\max_{\tau_{2\ell+1}^{\varepsilon_2} \leq t \leq \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \geq \varepsilon_1\}$  are independent.

*Proof of Lemma 12.1:* (a) It is fairly clear that, if  $\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}$ , then  $\tau_{2\ell}^{\varepsilon_2} < \tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1}$ , and  $\max_{\tau_{2\ell+1}^{\varepsilon_2} \leq t \leq \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \geq \|X(\tau_1^{\varepsilon_1})\| = \varepsilon_1$ .

Conversely, if  $\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}$ , then since  $\|X(t)\| \leq \varepsilon_2$  for  $t \in [\tau_{2\ell}^{\varepsilon_2}, \tau_{2\ell+1}^{\varepsilon_2}]$ , we have  $\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1}$ . On the other hand, if  $\max_{\tau_{2\ell+1}^{\varepsilon_2} \leq t \leq \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \geq \varepsilon_1$ , then  $\exists t \in (\tau_{2\ell+1}^{\varepsilon_2}, \tau_{2\ell+2}^{\varepsilon_2}) \subset (\tau_0^{\varepsilon_2}, \tau_{2\ell+2}^{\varepsilon_2}) = (\tau_0^{\varepsilon_1}, \tau_{2\ell+2}^{\varepsilon_2})$ , such that  $\|X(t)\| \geq \varepsilon_1$ . Thus  $\tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}$ , concluding the proof of Part (a) of Lemma 12.1.

(b) By Step 1, proof of Proposition 12.1,  $\{f(X(\tau_{2\ell+1}^{\varepsilon_2})) \in B\}$  is independent of  $\mathcal{F}^X(\tau_{2\ell}^{\varepsilon_2})$ . But  $\{\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}\} \in \mathcal{F}^X(\tau_{2\ell}^{\varepsilon_2})$ , so  $\{f(X(\tau_{2\ell+1}^{\varepsilon_2})) \in B\}$  and  $\{\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}\}$  are independent, and both belong to  $\mathcal{F}^X(\tau_{2\ell+1}^{\varepsilon_2})$ .

Let  $A_2 := \{\omega \in C[0, \infty) : \omega(\cdot) \text{ hits } \varepsilon_1 \text{ before hitting } 0 \text{ with } \omega(0) = \varepsilon_2\} \in \mathcal{B}(C[0, \infty))$ . Proposition 9.2 applied to  $\|X(\cdot)\|$  gives that

$$\mathbb{P}\left(\max_{\tau_{2\ell+1}^{\varepsilon_2} \leq t \leq \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \geq \varepsilon_1 \mid \mathcal{F}^X(\tau_{2\ell+1}^{\varepsilon_2})\right) = \mathbb{P}(\|X(\tau_{2\ell+1}^{\varepsilon_2} + \cdot)\| \in A_2 \mid \mathcal{F}^X(\tau_{2\ell+1}^{\varepsilon_2}))$$

equals  $\mathbb{P}(\|X(\tau_{2\ell+1}^{\varepsilon_2} + \cdot)\| \in A_2 \mid \|X(\tau_{2\ell+1}^{\varepsilon_2})\|)$ , a measurable function of  $\|X(\tau_{2\ell+1}^{\varepsilon_2})\|$ . But we have  $\|X(\tau_{2\ell+1}^{\varepsilon_2})\| \equiv \varepsilon_2$ , and therefore the event  $\{\max_{\tau_{2\ell+1}^{\varepsilon_2} \leq t \leq \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \geq \varepsilon_1\}$  is independent of  $\mathcal{F}^X(\tau_{2\ell+1}^{\varepsilon_2})$ . Combining this observation with the last paragraph, we complete the argument for Part (b).

This concludes the proof of Lemma 12.1.  $\square$

• With  $\mu$  defined as in Proposition 12.1, let  $\nu$  be the “angular measure” on  $([0, 2\pi), \mathcal{B}([0, 2\pi)))$  induced by  $\mu$  on  $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$ , through (2.9). Thus with  $\Theta(\cdot) := \arg(X(\cdot))$ , for every fixed  $\varepsilon > 0$  the random variables  $\{\Theta(\tau_{2\ell+1}^{\varepsilon})\}_{\ell \in \mathbb{N}_0}$  are I.I.D. with common distribution  $\nu$ , following Proposition 12.1.

We turn now to the proof of the FREIDLIN-SHEU formula for  $X(\cdot)$  in this setting: *For every function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  in the class  $\mathfrak{D}$ , defined as in subsection 4.1, we have*

$$\begin{aligned} g(X(\cdot)) &= g(x) + \int_0^\cdot \mathbf{1}_{\{\|X(t)\| > 0\}} \left( \mathbf{b}(\|X(t)\|) G'(X(t)) + \frac{1}{2} \mathbf{a}(\|X(t)\|) G''(X(t)) \right) dt \\ &\quad + \int_0^\cdot \mathbf{1}_{\{\|X(t)\| > 0\}} \sigma(\|X(t)\|) G'(X(t)) dW(t) + \left( \int_0^{2\pi} g'_\theta(0+) \nu(d\theta) \right) L^{\|X\|}(T). \end{aligned} \quad (12.2)$$

With the considerations at the start of this section and  $S(\cdot) := \|X(\cdot)\|$ , we can proceed exactly as the proof of Theorem 4.1, except for the step of proving (5.18), because now we do not have the help of (2.18).

We claim this convergence still holds. Setting  $N(T, \varepsilon) := \#\{\ell \in \mathbb{N} : \tau_{2\ell}^\varepsilon < T\}$  and  $h(\theta) := g'_\theta(0+)$ ,

$$\sum_{\{\ell : \tau_{2\ell+1}^\varepsilon < T\}} \varepsilon g'_{\Theta(\tau_{2\ell+1}^\varepsilon)}(0+) = \varepsilon N(T, \varepsilon) \cdot \frac{1}{N(T, \varepsilon)} \sum_{\ell=0}^{N(T, \varepsilon)-1} h(\Theta(\tau_{2\ell+1}^\varepsilon)) + O(\varepsilon).$$

First, we have  $\varepsilon N(T, \varepsilon) \xrightarrow[\varepsilon \downarrow 0]{} L^{\|X\|}(T)$  in probability, by Theorem VI.1.10 in REVUZ & YOR (1999).

Next, by the strong law of large numbers, we have  $\frac{1}{N} \sum_{\ell=0}^{N-1} h(\Theta(\tau_{2\ell+1}^\varepsilon)) \xrightarrow[N \rightarrow \infty]{} \int_0^{2\pi} h(\theta) \nu(d\theta)$ , a.e., for any fixed  $\varepsilon$ . By the definition of limit, we have  $\sup_{n \geq N} |(\frac{1}{n} \sum_{\ell=0}^{n-1} h(\Theta(\tau_{2\ell+1}^\varepsilon))) - \int_0^{2\pi} h(\theta) \nu(d\theta)| \xrightarrow[N \rightarrow \infty]{} 0$ , a.e., so this convergence is also valid in probability. Moreover, this convergence in probability is uniform in  $\varepsilon$ , because the distribution of the random variable  $\sup_{n \geq N} |(\frac{1}{n} \sum_{\ell=0}^{n-1} h(\Theta(\tau_{2\ell+1}^\varepsilon))) - \int_0^{2\pi} h(\theta) \nu(d\theta)|$  does not depend on  $\varepsilon$ . Now it is not hard to see that we have the convergence in probability

$$\frac{1}{N(T, \varepsilon)} \sum_{\ell=0}^{N(T, \varepsilon)-1} h(\Theta(\tau_{2\ell+1}^\varepsilon)) \xrightarrow[\varepsilon \downarrow 0]{} \int_0^{2\pi} h(\theta) \nu(d\theta), \quad \text{on the event } \left\{ N(T, \varepsilon) \xrightarrow[\varepsilon \downarrow 0]{} \infty \right\}.$$

Thus our claim holds on this event. On the complement of this event the terms  $\frac{1}{N(T, \varepsilon)} \sum_{\ell=0}^{N(T, \varepsilon)-1} h(\Theta(\tau_{2\ell+1}^\varepsilon))$  stay bounded, and we have  $\varepsilon N(T, \varepsilon) \xrightarrow[\varepsilon \downarrow 0]{} L^{\|X\|}(T) = 0$ , thus

$$\sum_{\{\ell : \tau_{2\ell+1}^\varepsilon < T\}} \varepsilon g'_{\Theta(\tau_{2\ell+1}^\varepsilon)}(0+) \xrightarrow[\varepsilon \downarrow 0]{} \left( \int_0^{2\pi} g'_\theta(0+) \nu(d\theta) \right) L^{\|X\|}(T) = 0, \quad \text{in probability.}$$

This establishes our claim, and obtains the FREIDLIN-SHEU Formula (12.2) for the state process  $X(\cdot)$  of the posited weak solution. With (12.2) just established, and (2.17) valid by assumption, we see that  $X(\cdot)$  generates a probability measure on  $(C[0, \infty)^2, \mathcal{B}(C[0, \infty)^2))$  which solves the local martingale problem associated with the triple  $(\sigma, \mathbf{b}, \mu)$ , where  $\mu$  is defined as in Proposition 12.1. This proves Part (ii) of Theorem 9.1, assuming that the stopping times  $\{\tau_m^\varepsilon\}_{m \in \mathbb{N}_0, \varepsilon > 0}$  are all finite with probability one.

**PROOF OF THEOREM 9.1(ii), PART B:** When the stopping times  $\{\tau_m^\varepsilon\}_{m \in \mathbb{N}_0, \varepsilon > 0}$  can be infinite, we proceed as follows.

*Step 1:* If  $\mathbb{P}(\tau_0^\varepsilon < \infty) = 0$ , then  $L^{\|X\|}(\cdot) \equiv 0$  and (12.2) holds for any  $\nu$ . Thus the conclusion of Part (ii) of Theorem 9.1 is true for any probability measure  $\mu$  on  $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$ . If  $\mathbb{P}(\tau_0^\varepsilon < \infty) > 0$ , we know from (2.17) that  $X(\cdot)$  can reach the origin and leave it with positive probability. So we can pick up a  $\varepsilon_0$  such that  $\mathbb{P}(\tau_1^{\varepsilon_0} < \infty) > 0$ . Then for every  $\varepsilon \in (0, \varepsilon_0]$ ,  $\ell \in \mathbb{N}_0$ , define the probability measure  $\mu_\ell^\varepsilon$  by

$$\mu_\ell^\varepsilon(B) := \frac{\mathbb{P}(\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon)) \mathbf{1}_{\{\tau_{2\ell+1}^\varepsilon < \infty\}} \in B)}{\mathbb{P}(\tau_{2\ell+1}^\varepsilon < \infty)}, \quad \forall B \in \mathcal{B}(\mathfrak{S}).$$

This is well-defined for  $\ell = 0$ , by our choice of  $\varepsilon_0$ . If  $\mathbb{P}(\tau_{2\ell+1}^\varepsilon < \infty) = 0$  for some  $\ell \geq 1$ , we redefine  $\mu_\ell^\varepsilon$  by  $\mu_\ell^\varepsilon := \mu_0^\varepsilon$ .

*Step 2:* It is straightforward but heavier in notation, to follow the steps of Proposition 12.1 and Lemma 12.1 and check that  $\mu_\ell^\varepsilon$  does not depend on either  $\varepsilon$  or  $\ell$ ; so we can define  $\mu := \mu_\ell^\varepsilon$ ,  $\forall \varepsilon > 0, \ell \in \mathbb{N}_0$ . Now we enlarge the original probability space by means of a countable collection of  $\mathfrak{S}$ -valued I.I.D. random variables  $\{\xi_\ell^\varepsilon\}_{\varepsilon \in \mathbb{Q}^+, \ell \in \mathbb{N}_0}$  with common distribution  $\mu$ , and independent of the  $\sigma$ -algebra  $\mathcal{F}$ . For every  $\varepsilon \in \mathbb{Q}^+$ ,  $\ell \in \mathbb{N}_0$ , we define the  $\mathfrak{S}$ -valued  $\tilde{f}(X(\tau_{2\ell+1}^\varepsilon)) := f(X(\tau_{2\ell+1}^\varepsilon)) \mathbf{1}_{\{\tau_{2\ell+1}^\varepsilon < \infty\}} + \xi_\ell^\varepsilon \mathbf{1}_{\{\tau_{2\ell+1}^\varepsilon = \infty\}}$ .

It is straightforward but tedious, to check that for any  $\varepsilon \in \mathbb{Q}^+$ , the random variables  $\{\tilde{f}(X(\tau_{2\ell+1}^\varepsilon))\}_{\ell \in \mathbb{N}_0}$  are independent with common distribution  $\mu$ . Then in the same way as in the last subsection, we can argue the convergence in (5.18) along rationals. The proof of Part (ii) of Theorem 9.1 is now complete.

**PROOF OF THEOREM 9.1(iii):** Finally, we argue Part (iii). Under the assumptions for Parts (ii) and (iii), let  $\mu$  be some probability measure for which the conclusion (ii) holds. Then by Proposition 6.1(b), we know that  $X(\cdot)$  also solves (6.4) with  $\gamma_i$  replaced by  $\int_{\mathfrak{S}} f_i(z) \mu(dz)$ . Thus we must have  $\gamma_i = \int_{\mathfrak{S}} f_i(z) \mu(dz)$ , which is (9.1), on the strength of  $\mathbb{P}(L^{\|X\|}(\infty) > 0) > 0$ . Moreover (2.18) also holds, namely

$$L^{R^A}(\cdot) \equiv \nu(A) L^{\|X\|}(\cdot), \quad \forall A \in \mathcal{B}([0, 2\pi)),$$

with  $R^A(\cdot) = \|X(\cdot)\| \cdot \mathbf{1}_A(\arg(X(\cdot)))$ . Thanks to  $\mathbb{P}(L^{\|X\|}(\infty) > 0) > 0$  again, we see from the above relationship that  $X(\cdot)$  uniquely determines  $\nu$ , thus also  $\mu$ . The proof of Theorem 9.1 is now complete.  $\square$

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